Normal-form Based Analysis of Climate Time Series

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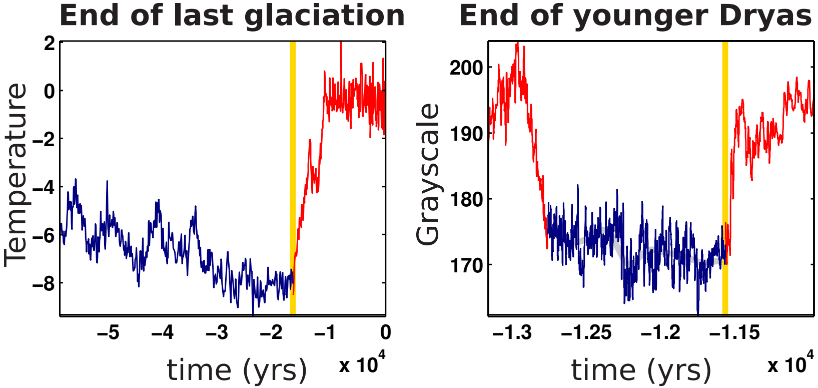
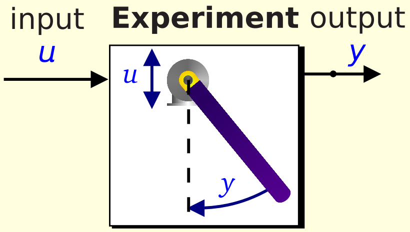
Outline

- Time series analysis
- Saddle-node induced tipping
- Estimate of normal form parameters from time series

Tipping in palaeoclimate time series



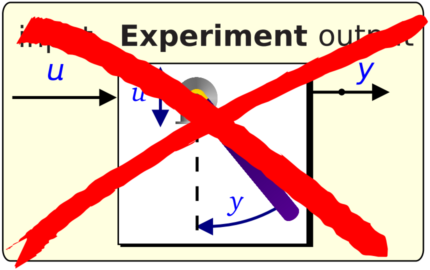
Background — time series analysis

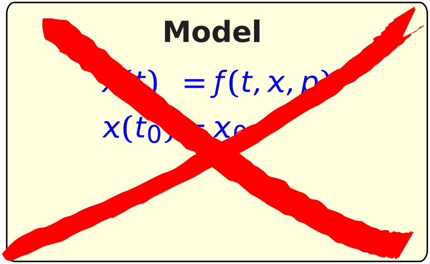


Model

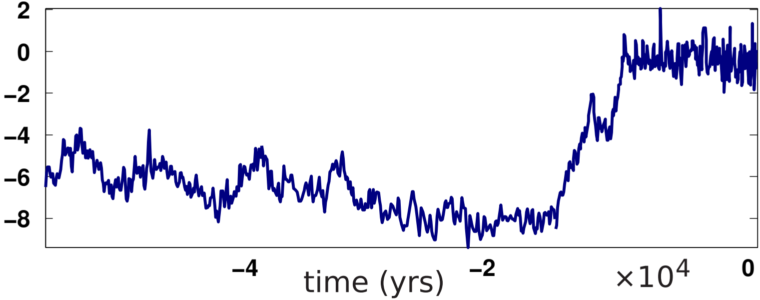
$$\dot{x}(t) = f(t, x, p)$$
$$x(t_0) = x_0$$

Background — time series analysis



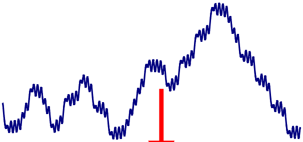


Observation

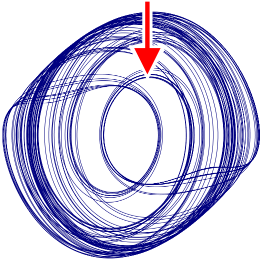


Background — time series analysis

Low-dim chaos & small noise

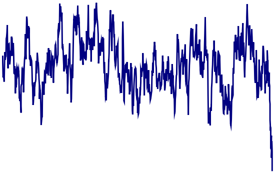


embedding, recurrence



[Kantz & Schreiber, 2000]

High-dim chaos or large noise



Assumption

- ▹ quasi-stationary

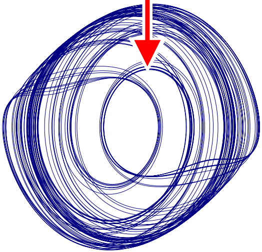
Questions

- ▹ finite t
- ▹ qualitative change of attractor

Background — time series analysis

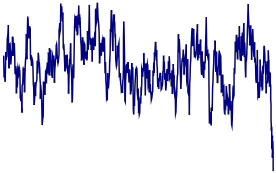
Low-dim chaos & small noise

embedding, recurrence



[Kantz & Schreiber, 2000]

High-dim chaos or large noise



Assumption

▹ quasi-stationary

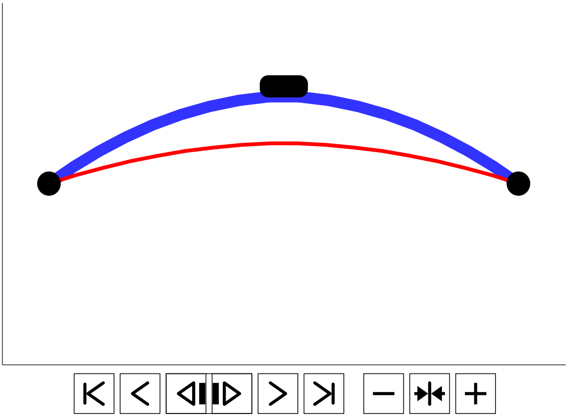
Questions

▹ finite *t*

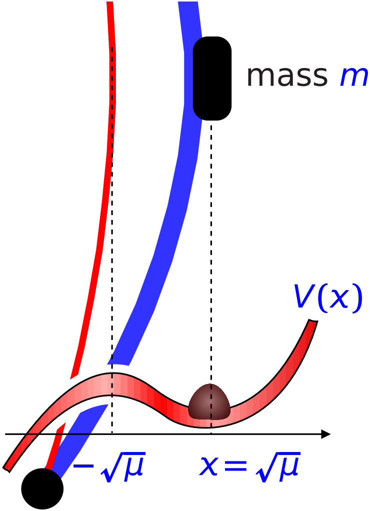
▹ qualitative change of attractor

Tipping — Mechanical caricature of positive feedback

- squishy beam, clamped and loaded with gradually increasing mass m



Tipping — Mechanical caricature of positive feedback



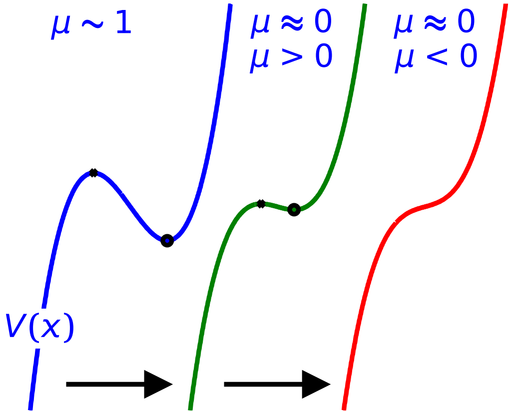
saddle-node normal form with drift and noise

$$\frac{\mathrm{d}}{\mathrm{d}t}x = -V'(x) = \mu - x^2 \textbf{ + noise}$$

$$\mu \sim m_{\text{critical}} - m$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\varepsilon$$

Tipping — Mechanical caricature of positive feedback



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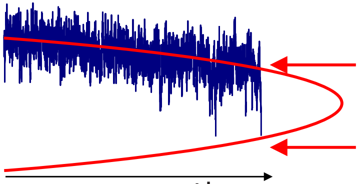
Estimate from time series

Approach to **Saddle-node** $\dot{x} = f(x, \mu) + \sigma\eta_t$

parameter $\dot{\mu} = -\varepsilon$



stable x_{eq}

unstable

time

Estimate from time series

Approach to **Saddle-node** $\dot{x} = f(x, \mu) + \sigma \eta_t$

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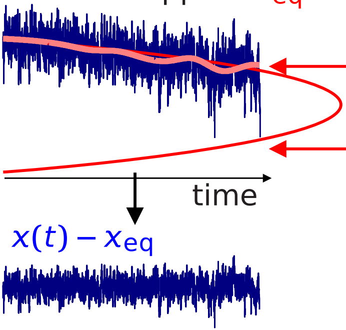


stable x_{eq}

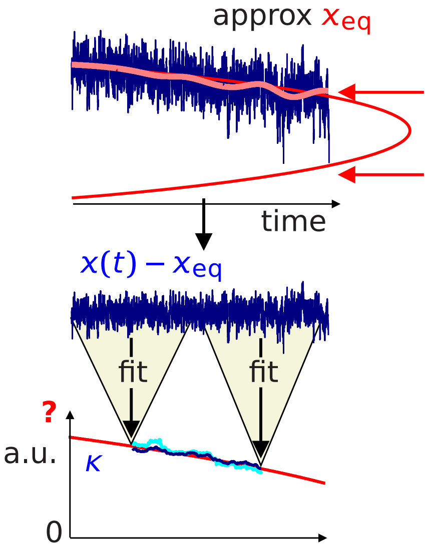
unstable

zero order

$\dot{x} = -(?)[x - x_{\mathrm{eq}}(\varepsilon t)] + ?$

Estimate from time series

Approach to **Saddle-node** $\dot{x} = f(x, \mu) + \sigma \eta_t$



parameter $\dot{\mu} = -\varepsilon$

stable x_{eq}

unstable

zero order

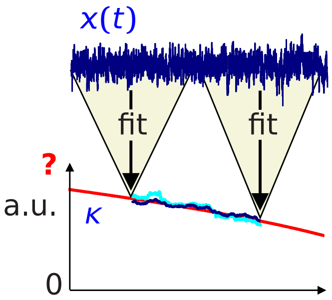
$\dot{x} = -(?)[x - x_{eq}(\varepsilon t)] + ?$

first order

$\dot{x} = -\kappa(\varepsilon t)[x - x_{eq}(\varepsilon t)] + ?$

AR(1)
DFA
Var

Estimate of linear decay rate



first order noise

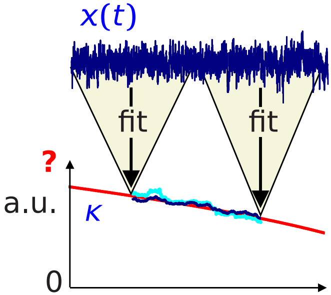
$\dot{x} = -\kappa(\varepsilon t)x + \sigma\eta_t$

AR(1) (Held&Kleinen'04)

DFA (Livina&Lenton'07)

Variance

Estimate of linear decay rate



first order

noise

$\dot{x} = -\ \kappa(\varepsilon t)x + \sigma\eta_t$

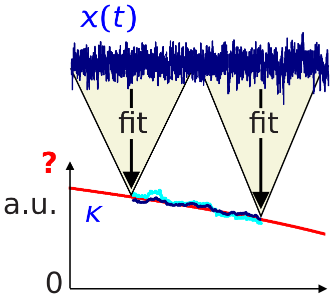
AR(1) (Held&Kleinen'04)

DFA (Livina&Lenton'07)

Variance

AR(1) $x(t_{n+1}) = \alpha\, x(t_n)$ $\Rightarrow$ fit α $\Rightarrow$ $\alpha = \exp(-\kappa\Delta_t)$

Estimate of linear decay rate



first order — noise

$$\dot{x} = -\ \kappa(\varepsilon t)x + \sigma\eta_t$$

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DFA (Livina&Lenton'07)
Variance

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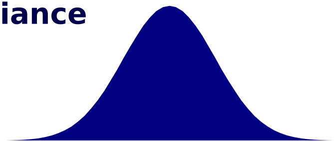
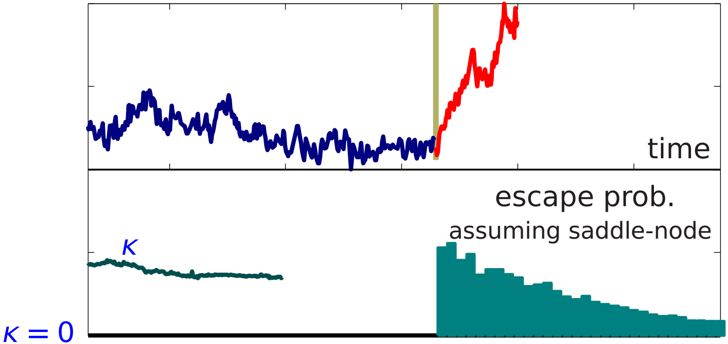
Variance

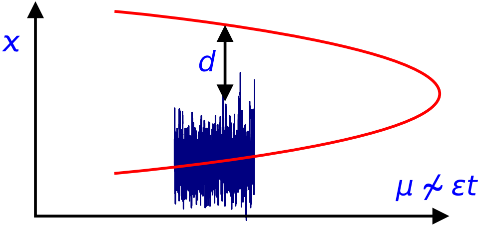


$\Rightarrow$ $\mathrm{Var} = \dfrac{\sigma^2}{\kappa}$

stationary distribution normal

When linear is not enough

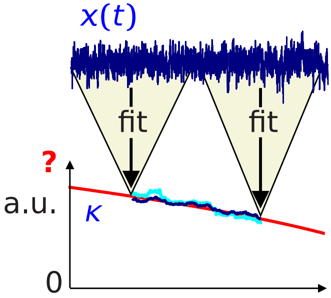




Need to know

κ and d

Back to estimates

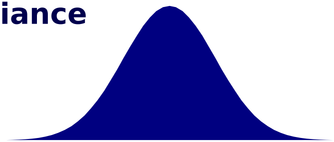


first order $\dot{x} = -\ \kappa(\varepsilon t)x + \sigma\eta_t$ ← noise

AR(1) (Held&Kleinen'04)
DFA (Livina&Lenton'07)
Variance

AR(1) $x(t_{n+1}) = \alpha\, x(t_n)$ $\Rightarrow$ fit α $\Rightarrow$ $\alpha = \exp(-\kappa\Delta_t)$

Variance



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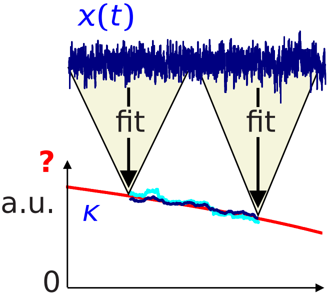
Back to estimates



$$\dot{x} = -\,\kappa(\varepsilon t)x + \mathbf{N}\,\mathbf{x}^2 + \sigma\eta_t$$

AR(1) (Held&Kleinen'04)

DFA (Livina&Lenton'07)

Variance

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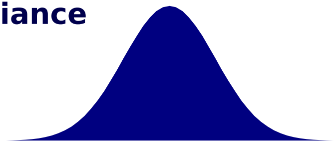
Variance



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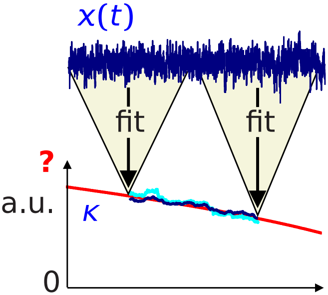
stationary distribution normal

Back to estimates



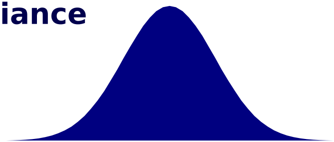
$$\dot{x} = -\,\kappa(\varepsilon t)x + \mathbf{N\,x^2} + \sigma\eta_t$$

AR(1) (Held&Kleinen'04)
DFA (Livina&Lenton'07)
Variance

AR(1) $x(t_{n+1}) = \alpha\, x(t_n)$

generalisation poor

Variance



$\Rightarrow \quad \mathrm{Var} = \dfrac{\sigma^2}{\kappa}$

stationary distribution normal

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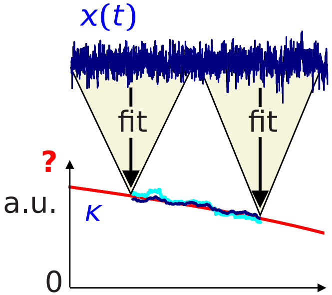

$$\dot{x} = -\ \kappa(\varepsilon t)x + \mathbf{N}\,\mathbf{x^2} + \sigma\eta_t$$

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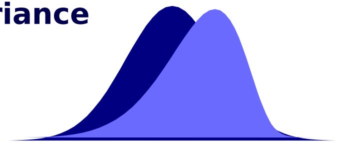

stationary distribution non-normal

Guttal

Livina,Kwasniok, Lenton'10

Estimates for nonlinear parts

Fokker-Planck equation Density p of

$$\dot{x} = f(x, \mu) + \sigma\eta_t$$

satisfies

$$\partial_t p = \frac{\sigma^2}{2}\partial_{xx} p - \partial_x \left[f(x, \mu)\, p \right]$$

Stationary density $p(x)$

$$\frac{1}{2}\partial_x p(x) = \sigma^{-2} f(x, \mu)\, p(x) + c$$

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Estimates for nonlinear parts

Fokker-Planck equation Density p of

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satisfies

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Stationary density $p(x)$

$$\frac{1}{2}\underbrace{\partial_x p(x)} = \sigma^{-2} f(x,\mu)\underbrace{p(x)} + c$$

empirical

Estimates for nonlinear parts

Fokker-Planck equation Density p of

$$\dot{x} = f(x,\mu) + \sigma \eta_t$$

satisfies

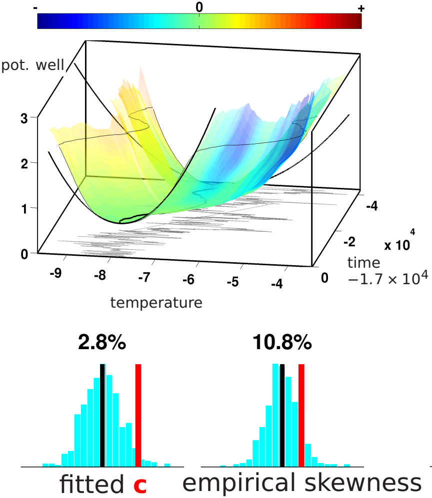
$$\partial_t p = \frac{\sigma^2}{2} \partial_{xx} p - \partial_x \left[f(x,\mu)\, p \right]$$

Stationary density $p(x)$

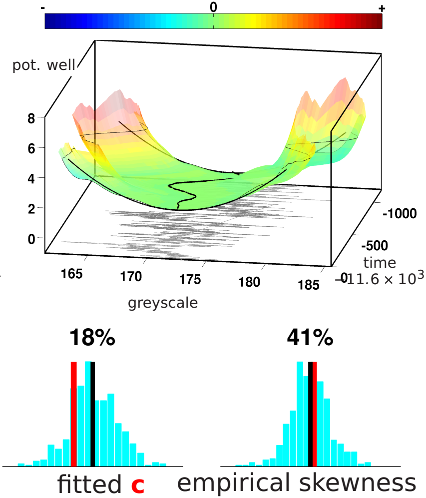
$$\frac{1}{2} \underbrace{\partial_x p(x)}_{\text{empirical}} = \overbrace{\sigma^{-2} f(x,\mu)}^{\text{linear estimate } -\kappa \mathbf{x}} \underbrace{p(x)}_{\text{empirical}} + \mathbf{c}$$

Paleo-climate records

End of last glaciation



End of Younger Dryas



Summary

- accuracy of estimates
  zero order $>$ first order $>$ second order
- but second order term necesssary to estimate tipping time/probability
- estimate tipping time/probability based on saddle-node normal form

[**JMTT,JS** on `arxiv`]

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