Improving the problem condition in control-based continuation experiments

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Summary. Simple mechanical prototype experiments have shown that it is possible to track (continue) branches of periodic orbits and equilibria regardless of their dynamical stability in physical experiments. The only prerequisite for continuation of unstable orbits is the presence of a tunable system parameter and a stabilizing feedback loop. This paper discusses problems that show up when continuing branches in ill-conditioned (or singularly perturbed) problems.

Background

Figure 1(a) shows a general schematic of the setup in which one can apply numerical continuation methods [4] in physical experiments [7]. Assume that one wants to learn about a branch of periodic orbits occuring in a nonlinear experiment that has a tunable system parameter p. For example, in [1, 2], which investigates a nonlinear 1dof oscillator subjected to harmonic forcing $f = a\sin(\omega t)$, the system parameter $p \in \mathbb{R}^2$ consists of the forcing frequency ω and the forcing amplitude a. In the schematic the black/grey parts indicate the parts that comprise the original, *uncontrolled*, experiment. Adding

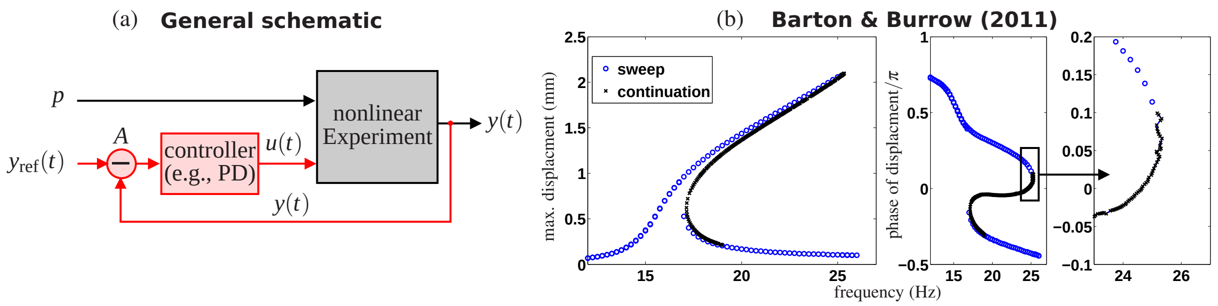

Figure 1: (a) Schematic of experiment with tunable system parameter p, control input u, output y and added feedback loop. (b) result of continuation of periodic orbits in a forced 1dof oscillator varying the frequency [1].

a feedback loop (shown in red in Figure 1(a)) then helps one to learn about unstable periodic orbits of the uncontrolled experiment in the following way:

1. set a periodic reference signal $y_{\mathrm{ref}}(t)$ as periodic input at point A and a system parameter p,
2. wait until the output $y(t)$ has settled to a periodic signal,
3. check the difference $F(y_{\mathrm{ref}}, p) = y(t) - y_{\mathrm{ref}}(t)$, which is a periodic signal.

If $F(y_{\mathrm{ref}}, p) = 0$ then $y_{\mathrm{ref}}(t)$ is the output corresponding to a periodic orbit of the uncontrolled experiment. This at first sight cumbersome approach permits one to find also dynamically unstable periodic orbits or periodic orbits that are too close to bifurcations to be accessible by classical parameter sweeps.
The main practical requirement on the setup is that the feedback loop (the red parts of Figure 1(a)) is *stabilizing* and *non-invasive*. More specifically, this means:

- (control is non-invasive) if $y_{\mathrm{ref}}(t) - y(t) = 0$ then $u(t) \to 0$ for $t \to \infty$.
- (control is stabilizing) the inputs y_{ref} and p determine locally uniquely the output y (after discarding transients), which must be periodic and have the same period as y_{ref}. Moreover, the output y depends smoothly on the inputs y_{ref} and p.

These two conditions imply that the map $F : (p, y_{\mathrm{ref}}) \mapsto y$ defined above is a well defined smooth map and that its roots are periodic orbits of the uncontrolled experiment. One can apply numerical root finding routines (that is, Newton iterations or quasi-Newton iterations [7]) and pseudo-arclength continuation to F to find periodic orbits of the uncontrolled experiment regardless of their dynamical stability. The results by [1] shown in Figure 1(b) were obtained by applying a Newton iteration only to the first Fourier coefficients of y_{ref} and correcting all other Fourier coefficients with a Pyragas type time-delayed difference scheme [6]. The control input u was not a separate input in [1]. Rather the input u was added to the harmonic forcing such that the experiment had a single (time-dependent) input $f = a\sin(\omega t) + \mathrm{PD}[y - y_{\mathrm{ref}}](t)$.

Improving the condition of the nonlinear problem

The main difficulty limiting the applicability of control-based continuation is, of course, the implementation of the stabilizing feedback loop. This problem is specific to each experiment and has to be solved anew in every application. The next limiting factor is that the map F, which is fed into the Newton iteration, can be evaluated only with low accuracy: the

leading Fourier coefficients of F are known with an accuracy $\varepsilon \sim \mathrm{tol}_{\mathrm{meas}}/\sqrt{N}$. In this estimate $\mathrm{tol}_{\mathrm{meas}}$ is the tolerance of the measurements of $y(t)$ and the adjustments of $u(t)$ (in [1] this is the accuracy with which the trajectory of the actuator driving the oscillator can be prescribed). The number N is the number of sampling points per period (assuming tlhat the errors in the measurements and adjustments are independent random fluctuations).
This implies that we cannot expect ε to be significantly smaller than 10^{-2} to 10^{-3} in mechanical experiments. The limited accuracy in the evaluation of F imposes a limit on the admissible condition of the Jacobian $\partial F(y_{\mathrm{ref}}, p)$ when solving for roots of F. In Figure 1(b) the zoom-in near the fold shows that for the experiment of [1] the uncertainty in the phase is already considerable. One likely source of this uncertainty is a bad (large) condition of $\partial F(y_{\mathrm{ref}}, p)$. The computational results in Figure 2 show the condition of $\partial F(y_{\mathrm{ref}}, p)$ for the prototypical weakly damped and weakly (hamornically) forced Duffing oscillator with hardening spring

$$\ddot{x} + \varepsilon\dot{x} + k_1 x + k_3 x^3 = a_c \cos(\omega t) + a_s \sin(\omega t), \tag{1}$$

where ε is small and a_c, $a_s \sim \varepsilon$. Panel (a) shows the amplitude and the phase (relative to the forcing) of the solution $(x, \dot{x})$

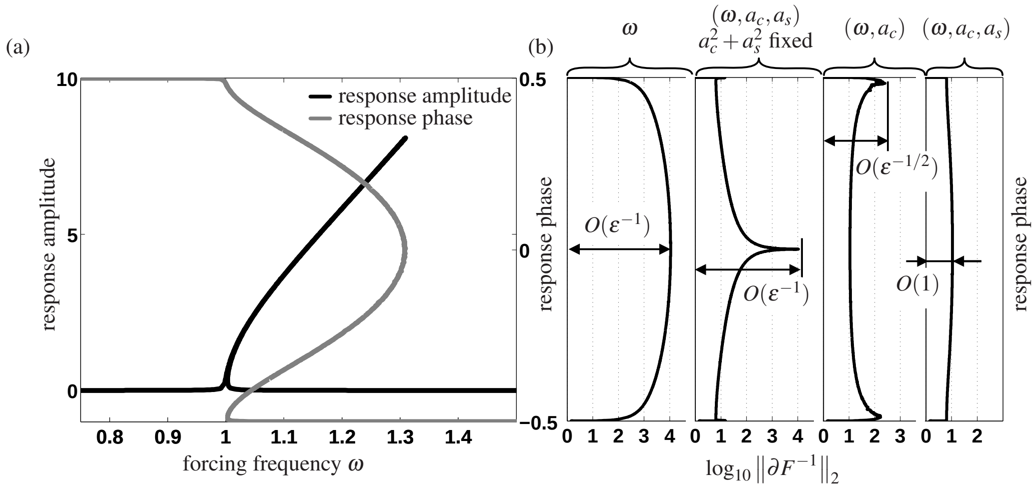

Figure 2: Linearized condition of nonlinear system for harmonically forced Duffing oscillator with $\varepsilon = 10^{-3}$, $k_1 = k_3 = (2\pi)^2$

depending the forcing frequency ω. Panel (b) shows $\|\partial F^{-1}\|_2$ for various extensions p. The y-axis matches the phase of the solution in panel (a), acting as a parameter along the resonance curve. The x-axis shows the norm of the inverse of ∂F in logarithmic scale ($\|\partial F\|_2$ is uniformly bounded). If p consists only of the primary continuation parameter ω then $\|\partial F^{-1}\|_2 \sim O(\varepsilon^{-1})$ along the entire resonance curve, If p consists of ω and the phase of the forcing (for example, by continuing in $p = (\omega, a_c, a_s)$ but keeping $\sqrt{a_c^2 + a_s^2}$ fixed) then $\|\partial F^{-1}\|_2$ approaches order ε^{-1} only in one point along the curve. If p consists of (ω, a_c) (that is, one continues in frequency and amplitude) then $\|\partial F^{-1}\|_2$ is at worst of order $\varepsilon^{-1/2}$. The worst condition occurs at the *base* of the resonance peak (for $\varepsilon = 0$ a pitchfork bifurcation occurs there, giving rise to this singularity). If p consists of (ω, a_c, a_s) then $\|\partial F^{-1}\|_2$ is bounded independent of ε. This provides a compelling case for using multi-parameter continuation techniques in experimental contexts to keep the condition of the nonlinear system small even if one is interested only in a submanifold of the solution manifold. The curves in Figure 2 have been obtained using basic simplex augmentation along the line (or cylinder) $\sqrt{a_c^2 + a_s^2} = 8\varepsilon$. General-purpose multi-parameter continuation codes such as Multifario ([5], to be linked to COCO [3]) are expected to provide a more robust solution and will be explored in the future.

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