

Extending the permissible control loop latency for the controlled inverted pendulum

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02 December, 2004

A pendulum can be stabilised in its upright position by proportional-plus-derivative (PD) feedback control only if the latency in the control loop is smaller than a certain critical delay. This critical delay is determined by the presence of a fully symmetric triple-zero eigenvalue singularity, a bifurcation of codimension three. We investigate three possible modifications of the PD scheme with the aim of extending the range of permissible delays. Effectively, these modifications introduce another parameter. This additional parameter can be used to continue the triple-zero singularity in four parameters until it gains a higher-order degeneracy imposing a new limit on the permissible delay. It turns out that the most effective modification is to feed back the value of the position with a small (intentional) additional delay on top of the control loop latency.

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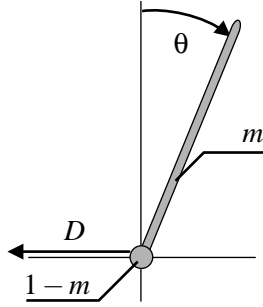


Figure 1: Sketch of the inverted pendulum on a cart.

1 Introduction

Latency in the feedback loop is often a critical issue in balancing tasks in robotics and biomechanics [7, 12]. For example, [4] reported on stick-balancing experiments that show how reflex time plays a crucial role in human balancing. The inverted planar pendulum on a cart is a simple prototype for these balancing tasks. In the setup depicted in figure 1 a feedback control force D is applied to the cart at the base of the pendulum to stabilise the pendulum in its upward position $\theta = 0$. This system is a standard example of a symmetric saddle equilibrium that is controlled to stability [5]. For instance, the textbook [10] on linear control theory uses the inverted pendulum as an illustrating example throughout.

The dynamics of the setup in figure 1 can be modelled by a second-order differential equation for the angular displacement θ of the tip of the pendulum, which can be written in dimensionless form as

$$\left(1 - \frac{3m}{4} \cos^2 \theta\right) \ddot{\theta} + \frac{3m}{8} \dot{\theta}^2 \sin(2\theta) - \sin \theta - D \cos \theta = 0. \quad (1)$$

Here m is the relative mass of the pendulum and $1 - m$ is the relative mass of the cart. Time t is measured in units of $\sqrt{2L/(3g)}$ where L is the length of the pendulum and g describes gravity. The rescaled state-dependent feedback control force D drives the cart in the horizontal direction trying to stabilise the upward equilibrium position $\theta = 0$.

If we take into account inherent delays in the feedback loop, D can only access the state of the system some time τ ago, that is, it has knowledge of $\theta(t - \tau)$ and $\dot{\theta}(t - \tau)$. Due to this delay in D , system (1) becomes a delay differential equation (DDE) for θ and $\dot{\theta}$ that has odd symmetry with respect to reflection in the origin. A conventional choice for the feedback force D is proportional-plus-derivative (PD) control; see, for example, [10] for the case without consideration of the delay. In the presence of delay, PD control takes the form

$$D = -a\theta(t - \tau) - b\dot{\theta}(t - \tau). \quad (2)$$

Here a and b are control gains and $\tau > 0$ is a fixed delay time, also referred to as the control loop latency.

The starting point of our paper is the common observation in previous investigations that there exists a critical delay $\tau = \tau_c$ beyond which stabilisation of the upward position $\theta = 0$ is impossible regardless of a and b . [16] investigated the dependence of (1)/(2) on all three parameters, in (a, b, τ) -space, and found that the equilibrium $\theta = 0$ can be stabilised linearly only for $\tau < \tau_c = \frac{1}{2}\sqrt{8 - 6m}$. At $\tau = \tau_c$ the island of permissible control gains shrinks to the point $(a, b) = (1, \tau_c)$ where the line of pitchfork bifurcations stabilizing the origin $\theta = 0$ and the curve of destabilising Hopf bifurcations become tangential to each other in the (a, b) -plane.

In [14] this singularity was identified as a non-semisimple triple-zero eigenvalue bifurcation in the presence of full reflection symmetry in the origin, a bifurcation of codimension three. This bifurcation has also been found in Chua's equation [1, 3, 9], which has the same \mathbb{Z}_2 -symmetry as (1)/(2). In [15] one finds a partial unfolding on the three-dimensional center manifold, a numerical bifurcation analysis, and how it links back to the DDE (1)/(2). This bifurcation analysis showed that there exist stable symmetric periodic orbits and chaotic attractors corresponding to small stable periodic and complex balancing motions of the pendulum tip in the vicinity of this singularity. The existence of stable periodic balancing motions could be verified experimentally by [11].

A direct consequence of the considerations in [15] is that this triple-zero eigenvalue bifurcation of the origin at $(a, b, \tau) = (1, \tau_c, \tau_c)$ is the limit for all stable small amplitude regimes of system (1)/(2). That is, accepting the small amplitude regimes found in [15] and [11] as successful balancing does not extend the range of permissible delays significantly. The reason is that small amplitude solutions have to be located in the local center manifold of the upward position. For $\tau > \tau_c$ the linearisation in the upward position has an eigenvalue with a positive real part of order one, which implies instability of the local center manifold and of all small amplitude solutions. Another conclusion in [14] is that all interesting dynamics are low-dimensional even though delay differential equations have an infinite-dimensional phase space.

The purpose of this paper is to investigate control schemes that extend PD control with the aim of increasing the critical delay τ_c beyond which successful control is not possible. In other words, the aim is to extend the permissible control loop latency. Specifically, we consider the following three natural extensions of PD control.

1. PMD control. If the control law cannot access the angular velocity, the proportional minus delay (PMD) controller

$$D = -a\theta(t - \tau_1) - b\dot{\theta}(t - \tau_2) \quad (3)$$

is a viable alternative to the PD scheme (2). Motivated by examples of biological systems, [2] has studied the special case $\tau_2 = 2\tau_1$ of (3), and also found a critical delay of $\tau_c = \frac{1}{2}\sqrt{4 - 3m}$. The PD control (2) corresponds in some sense to the limit $\tau_1 - \tau_2 \rightarrow 0$, and a suitable choice of a and b approaching ∞ . We study in section 2 how the critical delay τ_c , found by [2] for the special case, depends on all four parameters of (3).

2. Acceleration-dependent control. An apparent way to modify (2) is to take into account the angular acceleration if it is accessible and, hence, to consider

$$D = -a\theta(t - \tau) - b\dot{\theta}(t - \tau) - c\ddot{\theta}(t - \tau). \quad (4)$$

In this case the critical delay τ_c will depend also on the additional parameter c . Because system (1)/(4) is a neutral DDE, the requirement of stability of the essential spectrum imposes restrictions on c [8]. This case is studied in section 3.

3. Detuned PD control. An alternative modification of the PD control (2) is the introduction of a detuning in the delays between both arguments. This leads to

$$D = -a\theta(t - \tau_1) - b\dot{\theta}(t - \tau_2) \quad (5)$$

where either θ or $\dot{\theta}$ is fed back with an additional (possibly intentional) delay $|\tau_1 - \tau_2|$ on top of the control loop latency $\min\{\tau_1, \tau_2\}$. In section 4 we will investigate the maximal permissible control loop latency for this case.

Each of these control schemes introduces an additional parameter. Figure 2 illustrates how this affects stability charts in the (a, b) -plane. In all cases the region of stability (with 0 unstable directions of the origin) is bounded by curves of Hopf and pitchfork bifurcations that intersect at the points DZ and PH. Panel (a) shows the stability chart for the classical PD control (2) for $\tau = 1.25$, which is close to its critical delay of $\tau_c = \sqrt{2}$ (see also [16]) so that the region of stability is quite small. Panel (b) shows the stability chart for the PMD control law (3); note that the region of stability lies in a different part of the (a, b) -plane. Panels (c) and (d) show how the modification of PD control, by including acceleration dependence or detuning, respectively, is able to enlarge the region of stability for the same control loop latency of 1.25. Hence, we can expect that the associated introduction of the additional parameter will also change (and possibly increase) the value of the critical delay.

The basic idea is that the location of the codimension-three triple-zero eigenvalue singularity (where the stability region as shown in figure 2 shrinks to a point) and, hence, the value of the critical delay τ_c may change if we vary this additional parameter. It is now possible to continue this codimension-three singularity in four parameters until one meets a higher-order degeneracy imposing a new limit on the permissible delay. We find that the PMD control (3) has the classical PD control (2) as its optimal limiting case whereas the last two control schemes actually extend the overall permissible control loop latency.

2 PMD control

The PMD scheme using the feedback force (3) generalises the special case of the PMD controller studied in [2]. We can assume $\tau_2 > \tau_1$ without loss of generality since simple position feedback ($\tau_1 = \tau_2$) is unable to stabilise the origin [2, 14]. Thus, we may write

$$D = -a\theta(t - \tau) - b\theta(t - \tau - \delta) \quad (6)$$

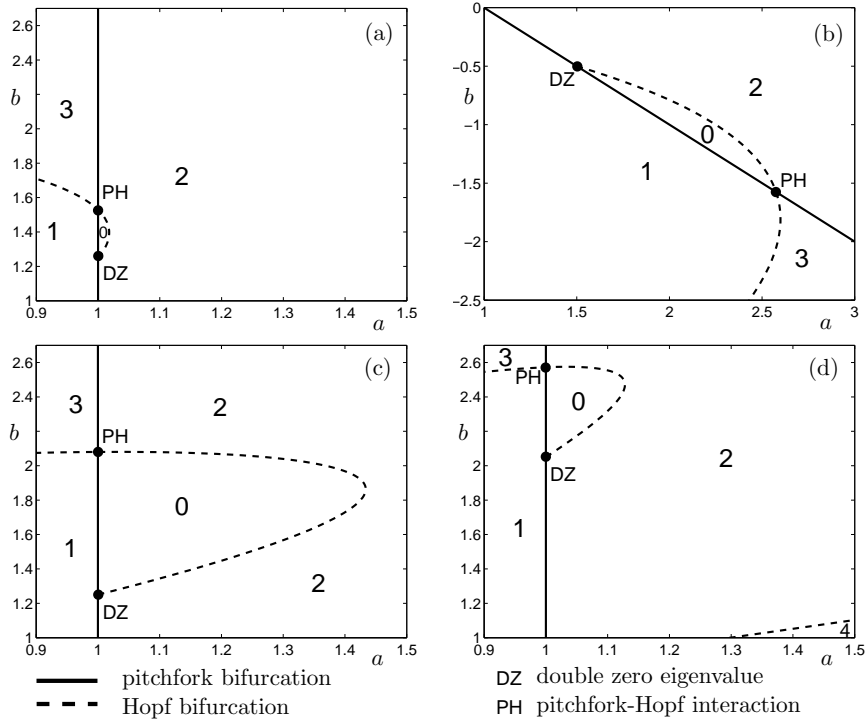


Figure 2: Stability charts in the (a, b) -plane with number of unstable directions of the origin for each region and codimension-one and -two bifurcations. Panel (a) is for the control law (2) with $\tau = 1.25$, panel (b) for (3) with $\tau_1 = 0.8$ and $\tau_2 = 1.8$, panel (c) for (4) with $\tau = 1.25$ and $c = 0.8$, and panel (d) for (5) with $\tau = 1.25$ and $\tau_2 = 2.05$; the mass ratio is $m = 0$ in all panels.

such that D depends on the four parameters (a, b, τ, δ) where $\delta > 0$. The delays in (6) have the following interpretation. The PMD controller feeds back the linear combination $a\theta(t) + b\theta(t - \delta)$ of the supposedly instantaneous position and the position at time δ ago. On top of that, the whole control loop has the latency τ . Our goal is to find a parameter set that stabilises the origin where the permissible control loop latency τ is as large as possible.

The linearisation of system (1) at the origin with feedback (6) has the characteristic function

$$\chi(\lambda) = \left(\lambda^2 (4 - 3m) - 4 + 4ae^{-\lambda\tau} + 4be^{-\lambda(\tau+\delta)} \right) / (4 - 3m). \quad (7)$$

Hence, $\lambda = 0$ is an eigenvalue if $a + b = 1$. System (1) undergoes a pitchfork bifurcation along this line in the (a, b) -plane (independent of the delays; see [2]). The geometric multiplicity of the eigenvalue 0 is always one, since the characteristic matrix $M(\lambda)$ has the form $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and, hence, rank one for $\lambda = 0$ along the line $a + b = 1$. The algebraic multiplicity of an eigenvalue corresponds to its multiplicity as a root of χ . Along the

curve

$$\begin{aligned} a(\omega) &= \left[1 + \omega^2 \left(1 - \frac{3}{4}m \right) \right] \frac{\sin(\omega(\tau + \delta))}{\sin(\omega\delta)}, \\ b(\omega) &= - \left[1 + \omega^2 \left(1 - \frac{3}{4}m \right) \right] \frac{\sin(\omega\tau)}{\sin(\omega\delta)} \end{aligned} \quad (8)$$

the linearisation has a complex pair of purely imaginary eigenvalues $\pm i\omega$. If this curve is crossed transversally in the (a, b) -plane system (1) undergoes a Hopf bifurcation. Consequently, the eigenvalue 0 of the linearisation of system (1) at the origin has algebraic multiplicity two at

$$a(0) = \frac{\tau + \delta}{\delta}, \quad b(0) = \frac{-\tau}{\delta}. \quad (9)$$

The Hopf curve (8) and the pitchfork line meet in $(a(0), b(0))$ in a non-semisimple double-zero eigenvalue bifurcation; see figure 2(b). If, in addition to (9),

$$\tau^2 + \tau\delta = 2 - \frac{3}{2}m \quad (10)$$

then the eigenvalue 0 has algebraic multiplicity three. This singularity, the non-semisimple triple-zero eigenvalue, has been studied in detail in [15]. In particular, if the triple-zero eigenvalue singularity is transversally stable (that is, all other roots of χ have a negative real part), a bounded island of linear stability of the origin exists in the (a, b) -plane for some τ and δ nearby. This island is limited by the pitchfork line and the Hopf curve (8) and shrinks to a point if the delays satisfy (10). Consequently, a PMD control using an argument with delay δ has the critical control loop latency

$$\tau_c = \frac{1}{2} \left[\sqrt{\delta^2 + 8 - 6m} - \delta \right] < \frac{1}{2} \sqrt{8 - 6m}.$$

The upper limit $\frac{1}{2}\sqrt{8 - 6m}$ can only be reached for δ approaching 0. That is, the smaller the delay δ in the delayed argument $b\theta(t - \tau - \delta)$ in the PMD control the larger the permissible control loop latency. The limit corresponds exactly to the critical delay for the PD control (2) where the velocity in (2) has been approximated by the finite difference $[\theta(t - \tau) - \theta(t - \tau - \delta)]/\delta$. In this sense one can view the PD control (2) as an optimal limiting case, with respect to the permissible control loop latency, of the general PMD control (3).

3 Acceleration-dependent control

If the angular acceleration can be measured, we may take it into account in the PD control (2), which leads to the control law (4). For D of this form, the location of the triple-zero eigenvalue singularity limiting the parameter range for stable small amplitude regimes depends on c .

System (1)/(4) is a neutral DDE. In order to fit it into the framework of the general theory for neutral equations as outlined in [8], we have to put it into the form

$$\frac{d}{dt} [d(x(t), x(t - \tau))] = f(x(t), x(t - \tau)). \quad (11)$$

The physical space of system (1)/(4) is \mathbb{R}^2 where the physical variable is $(\theta, \dot{\theta})$. We rescale the angular velocity $\dot{\theta}$ by the factor

$$\rho(\theta) = \frac{\cos \theta}{1 - \frac{3}{4}m \cos^2 \theta}$$

and introduce the new variable $x = (x_1, x_2)$ defined by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \rho(\theta)^{-1} \end{pmatrix} \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}.$$

This transforms (1)/(4) into form (11) where

$$\begin{aligned} d \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) &= \begin{pmatrix} x_1 \\ x_2 + c\rho(y_1)y_2 \end{pmatrix}, \\ f \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) &= \begin{pmatrix} \rho(x_1)x_2 \\ \tan x_1 + \rho(x_1)^2 x_2^2 \frac{\sin x_1}{\cos^2 x_1} - ay_1 - b\rho(y_1)y_2 \end{pmatrix}. \end{aligned} \quad (12)$$

A necessary condition for linear stability of the origin is that the essential spectrum of the time-shift semigroup for the difference equation

$$\frac{d}{dt} [\partial_1 d(0, 0)x(t) + \partial_2 d(0, 0)x(t - \tau)] = 0 \quad (13)$$

lies strictly within the unit circle. This criterion is called strong stability of the linear difference operator in [8]. In our case, it is equivalent to the condition that all roots of the characteristic function

$$\chi_0(\lambda) = 1 + e^{-\lambda\tau} \left(\frac{c}{1 - \frac{3}{4}m} \right)$$

of (13) lie in a strictly negative complex half-plane, giving that

$$|c| < 1 - \frac{3}{4}m. \quad (14)$$

Criterion (14) also implies that the essential spectrum of the time-shift semigroup of the linearisation of (11)/(12) depends continuously on the delay time τ [8]. Thus, while obeying restriction (14) on c , we can treat the neutral system (11)/(12) and its linearisation like a standard DDE. The origin is linearly stable if and only if all roots of the

characteristic function χ of the linearisation in the origin have negative real parts. The characteristic function χ has the form

$$\chi(\lambda) = \lambda^2 + \frac{1}{1 - \frac{3}{4}m} \left[e^{-\lambda\tau} (a + b\lambda + c\lambda^2) - 1 \right]. \quad (15)$$

Hence, $\lambda = 0$ is an eigenvalue if $a = 1$. This implies a pitchfork bifurcation along the line $a = 1$ in the (a, b) -plane (independent of c and τ). For $a = 1$ the geometric multiplicity of the eigenvalue 0 is always one, because the characteristic matrix

$$M(\lambda) = \lambda \left(\partial_1 d(0, 0) + e^{-\lambda\tau} \partial_2 d(0, 0) \right) - \left(\partial_1 f(0, 0) + e^{-\lambda\tau} \partial_2 f(0, 0) \right)$$

has the form $M(0) = -\rho(0) \begin{bmatrix} 0 & 1 \\ 0 & b \end{bmatrix}$, and, hence, rank one. The algebraic multiplicity of any eigenvalue corresponds to its multiplicity as a root of χ . If

$$\begin{aligned} a &= \cos(\omega\tau) \left[1 + \omega^2 \left(1 - \frac{3}{4}m \right) \right] + c\omega^2, \\ b &= \frac{\sin(\omega\tau)}{\omega} \left[1 + \omega^2 \left(1 - \frac{3}{4}m \right) \right] \end{aligned} \quad (16)$$

then the linearisation has a complex pair of purely imaginary eigenvalues $\pm i\omega$. If the curve defined by (16) is crossed transversally in the (a, b) -plane, system (11)/(12) undergoes a Hopf bifurcation. The eigenvalue 0 of the linearisation has algebraic multiplicity two if $\omega = 0$ in (16), that is, for $(a, b) = (1, \tau)$. The Hopf curve (16) and the line of pitchfork bifurcations meet each other in this point in the (a, b) -plane in a non-semisimple double-zero bifurcation. For small c and τ , there exists a bounded region of stability of the origin in the (a, b) -plane near $(a, b) = (1, \tau)$, bounded by the Hopf curve and the pitchfork line; see figure 2(c). This region shrinks to a point if

$$c = \frac{\tau^2}{2} - 1 + \frac{3}{4}m, \quad (17)$$

which corresponds to a non-semisimple triple-zero eigenvalue singularity. (This has been shown for $c = 0$ in [16, 14], and, hence, remains valid for small c .) That is, relation (17) defines the critical delay τ_c depending on the new free parameter c at least if c is small. Relation (17) allows us to enlarge the critical delay by increasing c . Stability criterion (14) is an upper limit implying that

$$\tau < \tau_c = \frac{\sqrt{2}}{2} \sqrt{8 - 6m}. \quad (18)$$

In other words, the maximal permissible delay is enlarged by a factor of $\sqrt{2}$ compared to the classical PD control (2). Figure 3(a) shows the roots of

$$\chi_{m=0}(\lambda) = \lambda^2 - 1 + e^{-\lambda\tau} (a + b\lambda + c\lambda^2) \quad (19)$$

with the largest real part for $m = 0$, computed with Newton's method (note that $\lambda = 0$ has multiplicity three in figure 3(a)). This is numerical evidence for the transversal stability of the triple-zero eigenvalue singularity for $m = 0$, c close to 1, the limit imposed by

the strong stability criterion (14) for $m = 0$, and (a, b, τ) satisfying the three conditions for the occurrence of the triple-zero eigenvalue $a = 1$, $b = \tau$ and (17). Moreover, we can construct the roots of χ for a general $m \in [0, 1]$ (as defined in (15)) from the roots of $\chi_{m=0}$. If $\chi_{m=0}(\lambda) = 0$ for the parameter set (a, b, c, τ) then

$$\chi\left(\frac{\lambda}{\sqrt{1 - \frac{3}{4}m}}\right) = 0 \quad \text{for} \quad \left(a, b\sqrt{1 - \frac{3}{4}m}, c\left[1 - \frac{3}{4}m\right], \tau\sqrt{1 - \frac{3}{4}m}\right)$$

for a general $m \in [0, 1]$. Thus, figure 3(a) implies transversal stability of the triple-zero eigenvalue singularity for all $m \in [0, 1]$ and c close to $1 - \frac{3}{4}m$.

Consequently, taking into account the angular acceleration $\ddot{\theta}$ improves the robustness of the control scheme with respect to delay by increasing the range of permissible control gains a and b . However, the dependence c of the feedback on $\dot{\theta}$ should only be weak since otherwise infinitely many modes of the linearisation become unstable. This restriction on c is imposed by inequality (14). Overall, the maximal permissible delay is increased by a factor of $\sqrt{2}$ compared to the PD control (2).

4 Detuned PD control

An alternative modification of the conventional PD scheme (2) is to assume two different delays as in (5). We think of this situation in terms of an extra delay, or detuning, δ on top of the inherent control loop latency τ and write the control law in the form

$$D = -a\theta(t - \tau - \delta) - b\dot{\theta}(t - \tau). \quad (20)$$

We do not know a-priori which argument should be detuned, so that δ could be positive or negative. However, the analysis in this section will show that $\delta > 0$ is the preferable choice. This means that the value of θ is fed back only after an additional delay time δ , instead of at the same time as the value of $\dot{\theta}$. In this section we investigate how one can tune the parameters a , b and δ such that the origin is linearly stable for τ as large as possible. The conventional PD scheme (2) is a special case corresponding to $\delta = 0$.

The linearisation of system (1) with (20) at the origin has the characteristic function

$$\chi(\lambda) = \lambda^2 + \frac{1}{1 - \frac{3}{4}m} \left(ae^{-\lambda(\tau+\delta)} + \lambda be^{-\lambda\tau} - 1 \right). \quad (21)$$

The characteristic matrix $M(\lambda)$ at $\lambda = 0$ is

$$M(0) = \frac{1}{4 - 3m} \begin{pmatrix} 0 & 4 - 3m \\ 4 - 4a & -4b \end{pmatrix}.$$

Hence, $\lambda = 0$ is an eigenvalue of geometric multiplicity one if $a = 1$. System (1) undergoes a pitchfork bifurcation along this line in the (a, b) -plane. Along the curve

$$\begin{aligned} a(\omega) &= \left[1 + \omega^2 \left(1 - \frac{3}{4}m \right) \right] \frac{\cos(\omega\tau)}{\cos(\omega\delta)}, \\ b(\omega) &= \left[1 + \omega^2 \left(1 - \frac{3}{4}m \right) \right] \frac{\sin(\omega(\tau + \delta))}{\omega \cos(\omega\delta)} \end{aligned} \quad (22)$$

the linearisation at the origin has a complex pair of purely imaginary eigenvalues $\pm i\omega$. If this curve is crossed transversally in the (a, b) -plane, system (1) undergoes a Hopf bifurcation. The Hopf curve (22) and the pitchfork line $a = 1$ meet in the point $(1, \tau + \delta)$ in the (a, b) -plane where the eigenvalue 0 has algebraic multiplicity two. This is a non-semisimple double-zero eigenvalue bifurcation of the origin; see figure 2(d). If

$$a = 1, \quad b = \tau + \delta, \quad \tau = \frac{1}{2}\sqrt{8 - 6m + 4\delta^2} \quad (23)$$

then the eigenvalue 0 has algebraic multiplicity three. The special case $\delta = 0$ (and $a = 1$, $b = \tau$) has been treated in [14] where the triple-zero eigenvalue singularity of the origin was found to be transversally stable. Consequently, this singularity will be transversally stable for δ close to 0 and, hence, τ slightly larger than $\frac{1}{2}\sqrt{8 - 6m}$. This implies that we can choose $\delta > 0$ and increase δ and τ further until the triple-zero singularity meets an additional degeneracy at

$$\begin{aligned} \tau = \tau_c &= \frac{1}{3}\sqrt{9 + 6\sqrt{3}} \cdot \frac{1}{2}\sqrt{8 - 6m} \approx 1.47 \cdot \frac{1}{2}\sqrt{8 - 6m}, \\ \delta = \delta_c &= (\sqrt{3} - 1) \cdot \tau_c \approx 1.07 \cdot \frac{1}{2}\sqrt{8 - 6m}. \end{aligned} \quad (24)$$

For these values of the parameters the origin has a quadruple-zero eigenvalue singularity. That is, the eigenvalue 0 has algebraic multiplicity four for $(a, b, \delta, \tau) = (1, \tau_c + \delta_c, \delta_c, \tau_c)$. Figure 3(b) shows the spectrum of the linearisation of the origin for this parameter constellation and $m = 0$, as computed from (1)/(20) with the package DDE-BIFTOOL [6]. This verifies numerically that the quadruple-zero eigenvalue singularity is transversally stable for $m = 0$. Moreover, if λ is a root for the parameter set (a, b, δ, τ) for $m = 0$ then

$$\chi\left(\frac{\lambda}{\sqrt{1 - \frac{3}{4}m}}\right) = 0 \quad \text{for} \quad \left(a, b\sqrt{1 - \frac{3}{4}m}, \delta\sqrt{1 - \frac{3}{4}m}, \tau\sqrt{1 - \frac{3}{4}m}\right)$$

for $m \in [0, 1]$. Consequently, we can find transversally stable triple-zero singularities for (a, b, δ, τ) in the vicinity of $(1, \tau_c + \delta_c, \delta_c, \tau_c)$ in the parameter space for all $m \in [0, 1]$ and, hence, regions of linear stability of the origin.

The locations (23) and (24) of the degeneracies imply that we can stabilise the origin using the detuned PD scheme for control loop latencies larger than $\frac{1}{2}\sqrt{8 - 6m}$ (the critical latency for the PD scheme) but less than τ_c , which is approximately 1.47 times $\frac{1}{2}\sqrt{8 - 6m}$. We can do so by choosing the parameters in the following way. We feed back the supposedly instantaneous value of $\dot{\theta}$, which is actually the value for time $t - \tau$ due to the latency τ . The value of θ is fed back with a deliberate delay δ smaller than but sufficiently close to δ_c . Furthermore, we choose the control gains a greater than but sufficiently close to 1 and b greater than but sufficiently close to $\tau + \delta$. This strategy works if the control loop latency is less than τ_c . It ensures that the parameters are in

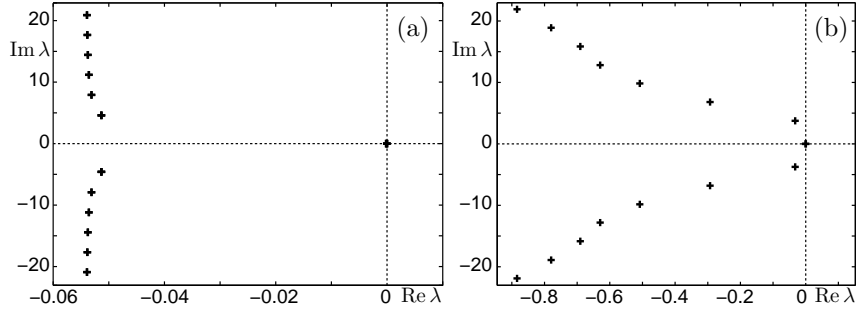


Figure 3: Dominant eigenvalues of the linearisation of system (1) in the origin. Panel (a) is for the acceleration dependent control (4) at $(a, b, c, m) = (1, \tau, 0.9, 0)$ and $\tau \approx 1.949$ satisfying (17), and panel (b) is for the detuned PD control (20) for $(a, b, m) = (1, \tau_1, 0)$, and $\tau_1 \approx 2.0759$ and $\tau_2 \approx 1.5197$ satisfying (23) and (24), respectively. The root 0 has multiplicity three in panel (a) and multiplicity four in panel (b).

the region of linear stability of the origin that exists nearby the triple-zero eigenvalue singularity [16, 14].

In an experiment where one can gradually increase the control loop latency (as was done in [11]) the detuning δ can be chosen with the following strategy. When increasing τ (keeping δ fixed) the equilibrium loses its stability in a supercritical Hopf bifurcation. When the onset of small amplitude oscillations is observed one can increase the detuning δ slightly to restabilise the origin. This procedure works as long as τ is less than τ_c , $b > \tau + \delta$, and a is sufficiently close to 1.

5 Conclusions

Delay in the control loop limits the ability of linear feedback controllers to stabilise even simple unstable systems like an inverted pendulum on a cart. Previous investigations of this classical example using a conventional PD control scheme have shown that there exists a critical delay beyond which stabilisation is impossible [16, 2, 14]. In this paper we have studied several ways to modify this control scheme by introducing an additional parameter. Two of these modifications actually extend the range of permissible delays.

The first option is to take into account the angular acceleration in the feedback. This extends the range of permissible delays at most by a factor of $\sqrt{2}$ compared to the conventional PD control scheme. The new limit on the permissible delay is imposed by the restriction that the feedback should depend only weakly on the acceleration. Otherwise infinitely many modes of the linearised system become unstable.

The second option is to feed back the angular position with a small (intentional) additional delay. This strategy increases the maximal permissible delay by a factor of approximately 1.47 compared to the PD scheme.

We find these critical delays by continuing a (fully symmetric) triple-zero eigenvalue

bifurcation, a codimension-three bifurcation, in four parameters until we meet a higher order degeneracy. This method exploits the fact that this bifurcation is the limit for all stable small amplitude regimes for all systems under consideration. Moreover, all results in [15] about the dynamics near the triple-zero eigenvalue bifurcation carry over to the systems studied in this paper. In particular, one should expect stable small amplitude periodic oscillations with arbitrarily large period and even small chaotic motions about the upside-down position.

We note that all options explored in this paper increase the maximal permissible delay in the control loop only in a limited way (compared to the classical PD control), that is, by a factor of less than 1.5. Control theory (see, for example, [13]) has proposed methods to compensate for arbitrarily large delays if the delay occurs only in the input. The problem of how to stabilise an inverted pendulum in the presence of arbitrarily large delays in an arbitrary position in the feedback loop remains open.

Acknowledgments

The research of J.S. is supported by EPSRC grant GR/R72020/01 and that of B.K. by an EPSRC Advanced Research Fellowship.

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