Journal of Mathematical Sciences, Vol. 124, No. 5, 2004

DYNAMICS OF MULTISECTION SEMICONDUCTOR LASERS

J. Sieber, L. Recke, and K. R. Schneider UDC 517.958; 517.95

ABSTRACT. We consider a mathematical model (the so-called traveling-wave system) which describes longitudinal dynamical effects in semiconductor lasers. This model consists of a linear hyperbolic system of PDEs, which is nonlinearly coupled with a slow subsystem of ODEs. We prove that a corresponding initial-boundary value problem is well posed and that it generates a smooth infinite-dimensional dynamical system. Exploiting the particular slow–fast structure, we derive conditions under which there exists a low-dimensional attracting invariant manifold. The flow on this invariant manifold is described by a system of ODEs. Mode approximations of that system are studied by means of bifurcation theory and numerical tools.

CONTENTS

1. Motivation

In commercial and private communication, the exchange of multimedia information grows rapidly. Thus, the corresponding data traffic increases exponentially and is characterized by a shift from voice communication to package-oriented data traffic. This presents a big challenge to strongly increase the data transmission rate. Due to their inherent speed, semiconductor lasers are of great interest as optical devices for fast data regeneration (reamplification, retiming, reshaping) in future photonic networks. Typically, these devices have a nonstationary working regime. As an example we mention the regime of high-frequency oscillations. Multisection lasers allow one to generate and to control such nonlinear effects by designing the longitudinal structure of the device (see, e.g., [13, 23]).

It is well known that the production of multisection semiconductor lasers is very expensive and time consuming. The goal of this paper is to demonstrate that mathematical models can be used to study the longitudinal dynamics of such lasers and to optimize their working regime.

Under some physical assumptions which can be verified experimentally we may focus on a special model describing the longitudinal dynamics of edge emitting multisection semiconductor lasers by the interaction of two physical variables: the electromagnetic field E, roughly speaking the light amplitude, and the effective carrier density n within the active zone of the device. The corresponding mathematical system has the structure

$$\begin{aligned} \frac{dE}{dt} &= H(n)E, \\ \frac{dn}{dt} &= \varepsilon(I - n - g(n)[E, E]). \end{aligned} \tag{1.1}$$

Translated from Sovremennaya Matematika. Fundamental'nye Napravleniya (Contemporary Mathematics. Fundamental Directions), Vol. 2, Differential and Functional Differential Equations. Part 2, 2003.

 1072–3374/04/1245–5298

Here, E is a complex vector depending on time t and on the one-dimensional spatial variable z characterizing the longitudinal direction of the laser, and n is a real vector depending only on time and describing the spatially section-wise averaged carrier density. Moreover, $H(n)$ is a first-order differential operator with respect to z. Hence, system (1.1) couples a linear system of partial differential equations (PDEs) for E with a system of ordinary differential equations (ODEs) for n. Furthermore, the variables E and n act on different time-scales implying a slow–fast structure of (1.1). This fact is expressed by the presence of the small parameter ε, which is the ratio between the averaged lifetime of a photon and the averaged lifetime of a carrier. Finally, g is a Hermitian form implying a symmetry of (1.1) with respect to rotation of the complex variable E.

The small parameter ε and the special structure of (1.1) permit one to formulate conditions guaranteeing the existence of a finite-dimensional invariant manifold such that the PDE–ODE model can be reduced to an ODE model. That way, the qualitative properties of the reduced model as a function of parameters can be studied by applying well-known continuation methods to determine the bifurcation diagram in the corresponding parameter plane numerically.

The paper is organized as follows: In the section "Modeling," we describe a special mathematical model, the so-called traveling-wave system (TWS), which represents a hyperbolic system of partial differential equations and of ordinary differential equations including initial and boundary conditions. In the section "Mathematical Analysis" we show that the corresponding initial-boundary value problem is well posed. In the section "Model Reduction," we exploit special properties and the special structure of the TWS in order to derive conditions guaranteeing that the TWS can be reduced to an ODE-system. This ODE-system can be approximated by a simplified ODE-system which can be interpreted physically as a system generated by finitely many modes. In the last section, "Mode Analysis," we present a numerical bifurcation analysis of the mode system. From that analysis it can be concluded that multisection semiconductor lasers can be designed in such a way to exhibit nonstationary working regimes.

2. Modeling

There is a hierarchy of models describing the behavior of semiconductor lasers ranging from the Maxwell–Bloch Equations to systems of delay-differential equations and simple rate equations [22]. In this paper, we focus on the *traveling-wave model*, which describes the effects in narrow longitudinally inhomogeneous laser diodes.

This model is a hyperbolic system of PDEs coupled with a system of ODEs [2, 11, 20]. It has been extended by adding polarization equations to include nonlinear gain of dispersion effects [1, 2, 5, 18]. In this section, we introduce the corresponding system of differential equations and specify the fundamental assumptions on its coefficients.

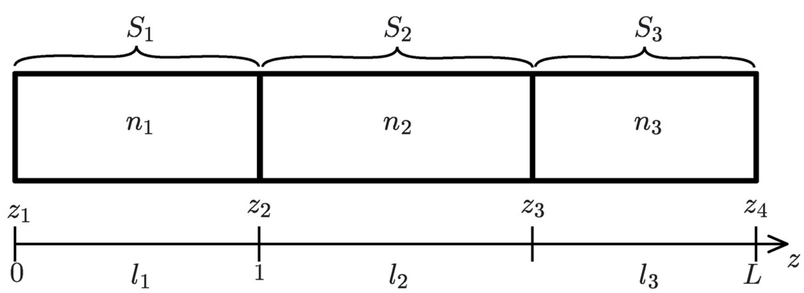

Fig. 1. Typical geometric configuration of the domain in a multisection laser with 3 sections. Here L is the length of the laser.

Let $\psi(t, z) \in \mathbb{C}^2$ describe the complex amplitude of the optical field split into a forward and a backward traveling wave. Let $p(t, z) \in \mathbb{C}^2$ be the corresponding nonlinear polarization. Both quantities depend on time and the one-dimensional spatial variable $z \in [0, L]$ (the longitudinal direction within the laser). The vector $n(t) \in \mathbb{R}^m$ represents the spatially averaged carrier densities within the individual sections of the laser (see Fig. 1).

The traveling-wave system consists of the traveling-wave equations

$$\partial_t \psi(t,z) = \sigma \partial_z \psi(t,z) + \beta(n(t),z)\psi(t,z) - i\kappa(z)\sigma_c \psi(t,z) + \rho(n(t),z)p(t,z), \tag{2.1}$$

$$\partial_t p(t,z) = (i\Omega_r(n(t),z) - \Gamma(n(t),z)) \cdot p(t,z) + \Gamma(n(t),z)\psi(t,z), \tag{2.2}$$

$$\begin{aligned} \frac{d}{dt} n_k(t) = I_k - \frac{n_k(t)}{\tau_k} - \frac{P}{l_k}\left(G_k(n_k(t)) - \rho_k(n_k(t))\right) \int_{S_k} \psi(t,z)^* \psi(t,z) dz \\ - \frac{P}{l_k} \rho_k(n_k(t)) \operatorname{Re}\left(\int_{S_k} \psi(t,z)^* p(t,z) dz \right) \quad \text{for } k = 1 \dots m \end{aligned} \tag{2.3}$$

accompanied by the inhomogeneous boundary conditions

$$\psi_1(t,0) = r_0 \psi_2(t,0) + \alpha(t), \quad \psi_2(t,L) = r_L \psi_1(t,L) \tag{2.4}$$

and the initial conditions

$$\psi(0,z) = \psi^0(z), \quad p(0,z) = p^0(z), \quad n(0) = n^0. \tag{2.5}$$

The Hermitian transpose of a $\mathbb{C}^2$-vector ψ is denoted by ψ^* in (2.3). We will define the appropriate function spaces and discuss the possible solution concepts in the section "Mathematical Analysis." The quantities and coefficients appearing above have the following physical meaning (see also Table 1 and Fig. 1):

	Typical range	Explanation
$\psi(t,z)$	$\mathbb{C}^2$	optical field, forward and backward traveling wave
$i \cdot p(t,z)$	$\mathbb{C}^2$	nonlinear polarization
$n_1(t)$	$(\underline{n}, \infty)$	spatially averaged carrier density in section S_1
$\operatorname{Im} \beta_k^0$	$\mathbb{R}$	frequency detuning
$\operatorname{Re} \beta_k^0$	< 0, $(-10, 0)$	decay rate due to internal losses
α_H	$(0, 10)$	negative of the line-width enhancement factor
g_1	≈ 1	differential gain in S_1
κ_k	$(-10, 10)$	real coupling coefficients for the optical field ψ
ρ_k	$[0, 1)$	maximum of the gain curve
Γ_k	$O(10^2)$	halfwidth of half-maximum of the gain curve
$\Omega_{r,k}$	$O(10)$	resonance frequency
I_k	$O(10^{-2})$	current injection
τ_k	$O(10^2)$	spontaneous lifetime for the carriers
P	$(0, \infty)$	scale of (ψ, p) (can be chosen arbitrarily)
r_0, r_L	$\mathbb{C}$, $\lvert r_0\rvert, \lvert r_L\rvert < 1$	facet reflectivities
$\alpha(t)$	$\mathbb{C}$	optical input at the facet $z = 0$

Table 1. Ranges and explanations of the variables and coefficients appearing in (2.1)–(3.3). See also [5, 18] to inspect their relations to the originally used physical quantities and scales.

The laser is subdivided into m sections S_k of length l_k with starting points z_k for $k = 1, \dots, m$. We scale the system such that $l_1 = 1$ and set $z_{m+1} = L$. Thus, $S_k = [z_k, z_{k+1}]$. All coefficients are assumed to be spatially constant in each section, i.e. if $z \in S_k$, $\kappa(z) = \kappa_k$, $\Gamma(n,z) = \Gamma_k(n_k)$, $\beta(n,z) = \beta_k(n_k)$, $\rho(n,z) = \rho_k(n_k)$. The matrices σ and σ_c are defined by

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The model for $\beta(n,z) = \beta_k(n_k) \in \mathbb{C}$ ($z \in S_k$) we use throughout the work reads

$$\beta_k(\nu) = d_k + (1 + i\alpha_{H,k})G_k(\nu) - \rho_k(\nu),$$

where $d_k \in \mathbb{C}$, $\alpha_{H,k} \in \mathbb{R}$, and $\operatorname{Re} d_k < 0$. A section S_k is either *passive* (then the functions G_k and ρ_k are identically zero) or *active*. In this case, $G_k : (\underline{n}, \infty) \to \mathbb{R}$ is a smooth[1] strictly monotone increasing function satisfying $G_k(1) = 0$, $G_k'(1) > 0$. Its limits are

$$\lim_{\nu \searrow \underline{n}} G_k(\nu) = -\infty, \quad \lim_{\nu \to \infty} G_k(\nu) = \infty, \quad \text{where} \quad \underline{n} \le 0.$$

Typical models for G_k in active sections are

$$\begin{aligned} G_k(\nu) &= \tilde{g}_k \log \nu \qquad (\underline{n} = \ 0) \quad \text{or} \\ G_k(\nu) &= \tilde{g}_k \cdot (\nu - 1) \quad (\underline{n} = \ -\infty). \end{aligned}$$

If $G_k \not\equiv 0$, the function $\rho(n,z) = \rho_k(n_k)$ is bounded for $n_k < 1$. Moreover, we suppose

$$\rho_k,\ \Omega_{r,k},\ \Gamma_k :\ (\underline{n}, \infty) \to \mathbb{R}$$

to be smooth and Lipschitz continuous and $\Gamma_k(\nu) > 1$.

The coefficients r_0 and r_L in (2.4) are complex with modulus less than 1. The inhomogeneity $\alpha(t)$ is bounded but may be discontinuous in time. The variables and coefficients, their physical meanings, and their typical ranges are shown in Table 1.

Finally, we introduce the Hermitian form

$$g_k(\nu)\left[\begin{pmatrix}\psi\\ p\end{pmatrix}, \begin{pmatrix}\varphi\\ q\end{pmatrix}\right] = \frac{1}{l_k}\int_{S_k} (\psi^*(z), p^*(z)) \begin{pmatrix} G_k(\nu) - \rho_k(\nu) & \frac{1}{2}\rho_k(\nu) \\ \frac{1}{2}\rho_k(\nu) & 0 \end{pmatrix} \begin{pmatrix}\varphi(z)\\ q(z)\end{pmatrix} dz \tag{2.6}$$

and the notation

$$\begin{aligned} \|\psi\|_k^2 &= \int_{S_k} \psi^*(z)\psi(z)dz, \\ f_k(\nu, (\psi, p)) &= I_k - \frac{\nu}{\tau_k} - P g_k(\nu)\left[\begin{pmatrix}\psi\\ p\end{pmatrix}, \begin{pmatrix}\psi\\ p\end{pmatrix}\right] \end{aligned} \tag{2.7}$$

for $\nu \in [\underline{n}, \infty)$ and $\psi, p \in \mathbb{L}^2([0,L]; \mathbb{C}^2)$. Using this notation, (2.3) reads as

$$\frac{dn_k}{dt} = f_k(n_k, (\psi, p)) \text{ for } k = 1, \ldots, m. \tag{2.8}$$

3. Mathematical Analysis

In this section, we treat the inhomogeneous initial-boundary value problem (2.1)–(2.4) as an autonomous nonlinear evolution equation

$$\frac{du}{dt} = Au + g(u), \quad u(0) = u_0, \tag{3.1}$$

where $u(t)$ is an element of a Hilbert space V, A is the generator of a C_0 semigroup $S(t)$, and $g : U \subseteq V \to V$ is smooth and locally Lipschitz-continuous in an open set $U \subseteq V$. The inhomogeneity in (2.4) is included in (3.1) as a component of u.

[1]The notation *smooth* refers to C^∞ throughout this paper.

3.1. Notation. The Hilbert space V is defined by

$$V := \mathbb{L}^2([0,L];\mathbb{C}^4) \times \mathbb{R}^m \times \mathbb{L}^2_\eta([0,\infty);\mathbb{C}), \tag{3.2}$$

where $\mathbb{L}^2_\eta([0,\infty);\mathbb{C})$ is the space of weighted square-integrable functions. The scalar product of $\mathbb{L}^2_\eta([0,\infty);\mathbb{C})$ is defined by

$$(v,w)_\eta := \operatorname{Re}\int_0^\infty \bar{v}(x)\cdot w(x)(1+x^2)^\eta dx.$$

We choose $\eta < -1/2$ such that the space $\mathbb{L}^\infty([0,\infty);\mathbb{C})$ is continuously embedded in $\mathbb{L}^2_\eta([0,\infty);\mathbb{C})$. The complex plane is treated as a two-dimensional real plane in the definition of the vector space V such that the standard $\mathbb{L}^2$ scalar product $(\cdot,\cdot)_V$ of V is differentiable. The corresponding components of $v \in V$ are denoted by

$$v = (\psi_1, \psi_2, p_1, p_2, n, a).$$

The spatial variable in ψ and p is denoted by $z \in [0,L]$, whereas the spatial variable in a is denoted by $x \in [0,\infty)$. The Hilbert space $\mathbb{H}^1_\eta([0,\infty);\mathbb{C})$ equipped with the scalar product

$$(v,w)_{1,\eta} := (v,w)_\eta + (\partial_x v, \partial_x w)_\eta$$

is densely and continuously embedded in $\mathbb{L}^2_\eta([0,\infty);\mathbb{C})$. Moreover, its elements are continuous [19]. Consequently, the Hilbert spaces

$$W := \mathbb{H}^1([0,L];\mathbb{C}^2) \times \mathbb{L}^2([0,L];\mathbb{C}^2) \times \mathbb{R}^m \times \mathbb{H}^1_\eta([0,\infty);\mathbb{C}),$$
$$W_{\mathrm{BC}} := \{(\psi,p,n,a) \in W : \psi_1(0) = r_0\psi_2(0) + a(0), \psi_2(L) = r_L\psi_1(L)\}$$

are densely and continuously embedded in V. The linear functionals $\psi_1(0) - r_0\psi_2(0) - a(0)$ and $\psi_2(L) - r_L\psi_1(L)$ are continuous on W. We define the linear operator $A : W_{\mathrm{BC}} \to V$ by

$$A(\psi_1,\psi_2,p,n,a) := (-\partial_z\psi_1, \partial_z\psi_2, 0, 0, \partial_x a).$$

The definitions of A and W_{BC} treat the inhomogeneity α in the boundary condition (2.4) as the boundary value at 0 of the variable a. We define the open set $U \subseteq V$ by

$$U := \{(\psi,p,n,a) \in V : n_k > \underline{n} \text{ for } k = 1,\dots,m\}$$

and the nonlinear function $g : U \to V$ by

$$g(\psi,p,n,a) := \begin{pmatrix} \beta(n)\psi - i\kappa\sigma_c\psi + \rho(n)p \\ (i\Omega_r(n) - \Gamma(n))p + \Gamma(n)\psi \\ (f_k(n_k,(\psi,p)))_{k=1}^m \\ 0 \end{pmatrix}. \tag{3.3}$$

The corresponding coefficients of (2.1)–(2.3) define the smooth maps $\beta : (\underline{n},\infty)^m \to \mathcal{L}(\mathbb{L}^2([0,L];\mathbb{C}^2))$ and $\rho, \Omega_r, \Gamma : \mathbb{R}^m \to \mathcal{L}(\mathbb{L}^2([0,L];\mathbb{C}^2))$. The function g is continuously differentiable to any order with respect to all arguments and its Frechet derivative is bounded in any closed bounded ball $B \subset U$.

In the sequel the theory of C_0-semigroups and the concepts of classical and mild solutions [12] to (3.1) play a crucial role.

Lemma 1. *$A : W_{\mathrm{BC}} \subset V \to V$ generates a C_0-semigroup $S(t)$ of bounded operators in V.*

The inhomogeneous initial-boundary value problem (2.1)–(2.5) and the autonomous evolution system (3.1) are equivalent in the following sense: Suppose $\alpha \in \mathbb{H}^1([0,T);\mathbb{C})$ in (2.4). Let $u = (\psi,p,n,a)$ be a classical solution of (3.1). Then u satisfies (2.1)–(2.2) and (2.5) in $\mathbb{L}^2$ and (2.3), (2.4) for each $t \in [0,T]$ if and only if $a^0|_{[0,T]} = \alpha$. On the other hand, assume that (ψ,p,n) satisfies (2.1)–(2.2) and (2.5) in $\mathbb{L}^2$ and (2.3), (2.4) for each $t \in [0,T]$. Then, we can choose a $a^0 \in \mathbb{H}^1_\eta([0,\infty);\mathbb{C})$ such that $a^0|_{[0,T]} = \alpha$ and obtain that $u(t) = (\psi(t), p(t), n(t), a^0(t+\cdot))$ is a classical solution of (3.1) in $[0,T]$.

Mild solutions of (3.1) are a reasonable generalization of the classical solution concept of (2.1)–(2.4) to boundary conditions including discontinuous inputs $\alpha \in \mathbb{L}^2_\eta([0,\infty);\mathbb{C})$.

3.2. Global existence and uniqueness of solutions. In order to prove uniqueness and global existence of solutions to (3.1), we introduce the so-called truncated problem:

For any bounded ball $B \subset U$ which is closed w.r.t. V, we choose $g_B : V \to V$ such that g_B is smooth and globally Lipschitz continuous and $g_B(u) = g(u)$ for all $u \in B$. This is possible because the Frechet derivative of g is bounded in B and the scalar product in V is differentiable with respect to its arguments. We call

$$\frac{du}{dt} = Au + g_B(u), \quad u(0) = u_0 \tag{3.4}$$

the truncated problem to (3.1). The following lemma is a consequence of the results in [12].

Lemma 2 (global existence for the truncated problem)**.** *The truncated problem* (3.4) *has a unique global mild solution* $u(t)$ *for any* $u_0 \in V$*. If* $u_0 \in W_{\mathrm{BC}}$*,* $u(t)$ *is a classical solution of* (3.4)*.*

Corollary 3 (local existence)**.** *Let* $u_0 \in U$*. There exists a* $t_{\mathrm{loc}} > 0$ *such that the evolution problem* (3.1) *has a unique mild solution* $u(t)$ *on the interval* $[0, t_{\mathrm{loc}}]$*. If* $u_0 \in W_{\mathrm{BC}} \cap U$*,* $u(t)$ *is a classical solution of* (3.1) *in* $[0, t_{\mathrm{loc}}]$*.*

In order to extend the result of Lemma 2 to the evolution equation (3.1), we need the following a priori estimate for the solutions of the truncated problem (3.4).

Lemma 4. *Let* $T > 0$ *and* $u_0 \in W_{\mathrm{BC}} \cap U$*. If* $\underline{n} > -\infty$*, we suppose that* $I_k\tau_k > \underline{n}$ *for all* $k = 1, \ldots, m$*. There exists a closed bounded ball* B *such that* $B \subset U$ *and the solution* $u(t)$ *of the* B*-truncated problem* (3.4) *starting at* u_0 *stays in* B *for all* $t \in [0, T]$*.*

Moreover, a solution $u(t)$ starting at $u_0 \in W_{\mathrm{BC}} \cap U$ and staying in a bounded closed ball $B \subset U$ in $[0, T]$ is a classical solution in the whole interval $[0, T]$ because of the structure of the nonlinearity g.

Lemma 4 implies

Theorem 5 (global existence and uniqueness)**.** *Suppose that* $T > 0$*,* $u_0 = (\psi^0, p^0, n^0, a^0) \in U$*, and* $\|a^0|_{[0,T]}\|_\infty < \infty$*. If* $\underline{n} > -\infty$*, let* $I_k\tau_k > \underline{n}$ *for all* $k = 1, \ldots, m$*. There exists a unique mild solution* $u(t)$ *of* (3.1) *in* $[0, T]$*. Furthermore, if* $u_0 \in W_{\mathrm{BC}} \cap U$*,* $u(t)$ *is a classical solution of* (3.1)*.*

Corollary 6 (global boundedness)**.** *Let* $u_0 = (\psi^0, p^0, n^0, a^0) \in U$ *and* $\|a^0\|_\infty < \infty$*. There exists a constant* C *such that* $\|u(t)\|_V \le C$*.*

The following corollary is an immediate consequence of Theorem 5 and the general theory of C_0-semigroups [12] (see [17] for details).

Corollary 7 (smooth semiflow)**.** *The nonlinear equation* (3.1) *defines a semiflow* $S(t; u_0)$ *for* $t > 0$ *which is strongly continuous in* t *and smooth in* u_0 *and in all parameters.*

4. Model Reduction

4.1. Introduction of a small parameter. In the following we restrict ourselves to system (2.1)–(2.3) with the homogeneous boundary conditions (b.c.)

$$\psi_1(t, 0) = r_0\psi_2(t, 0), \quad \psi_2(t, L) = r_L\psi_1(t, L). \tag{4.1}$$

We reformulate (2.1)–(2.3) to exploit its particular structure. The space-dependent subsystem is linear in ψ and p and can be rewritten in the form

$$\partial_t \begin{pmatrix} \psi \\ p \end{pmatrix} = H(n) \begin{pmatrix} \psi \\ p \end{pmatrix}, \tag{4.2}$$

where the linear operator

$$H(n) = \begin{pmatrix} \sigma\partial_z + \beta(n) - i\kappa\sigma_c & \rho(n) \\ \Gamma(n) & i\Omega_r(n) - \Gamma(n) \end{pmatrix} \tag{4.3}$$

acts from

$$Y := \{(\psi, p) \in \mathbb{H}^1([0, L]; \mathbb{C}^2) \times \mathbb{L}^2([0, L]; \mathbb{C}^2) : \psi \text{ satisfying (4.1)}\}$$

into $X = \mathbb{L}^2([0, L]; \mathbb{C}^4)$. $H(n)$ generates a C_0-semigroup $T_n(t)$ acting in X. It can be proven that $T_n(t)$ is eventually differentiable (for the definition see, e.g., [10, 12]) in the case $r_0 r_1 = 0$. The coefficients κ, and, for each $n \in \mathbb{R}^m$, $\beta(n)$, $\Omega_r(n)$, $\Gamma(n)$, and $\rho(n)$ are linear operators in $\mathbb{L}^2([0, L]; \mathbb{C}^2)$ defined by the corresponding coefficients in (2.1), (2.2). The maps $\beta, \rho, \Gamma, \Omega_r : \mathbb{R}^m \to \mathcal{L}(\mathbb{L}^2([0, L]; \mathbb{C}^2))$ are smooth.

We observe that I_k and τ_k^{-1} in (2.7) are approximately two orders of magnitude smaller than 1 (see Table 1). Hence, we can introduce a small parameter ε and set $P = \varepsilon$ in (2.3) such that (2.8) reads

$$\frac{dn_k}{dt} = f_k(n_k, E) = \varepsilon(F_k(n_k) - g_k(n_k)[E, E]) \tag{4.4}$$

for $E \in X$, where the coefficients in $F_k(n_k) = \varepsilon^{-1}(I_k - n_k \tau_k^{-1})$ are of order 1. Although ε is not directly accessible, we treat it as a parameter and consider the limit as ε tends to 0 while keeping F_k fixed. For $\varepsilon = 0$, the carrier density n is stationary, that is, it enters the linear subsystem (4.2) as a parameter. Now, we will investigate the longtime behavior of this linear equation, where, for brevity, we omit the argument n.

4.2. Spectral Properties of $H(n)$. In this subsection, we investigate the spectrum of the operator $H(n)$, treating n as a parameter.

First we define the set of complex "resonance frequencies"

$$\mathcal{W} := \{c \in \mathbb{C} : c = i\Omega_{r,k} - \Gamma_k \text{ for at least one } k \in \{1, \dots, m\}\} \subset \mathbb{C}$$

and the function $\chi : \mathbb{C} \setminus \mathcal{W} \to \mathcal{L}(\mathbb{L}^2([0, L]; \mathbb{C}^2))$ by

$$\chi(\lambda) := \frac{\rho\Gamma}{\lambda - i\Omega_r + \Gamma} \in \mathcal{L}(\mathbb{L}^2([0, L]; \mathbb{C}^2)) \text{ for each } \lambda \in \mathbb{C} \setminus \mathcal{W}$$

(see [5, 18] for details). For $\lambda \in \mathbb{C} \setminus \mathcal{W}$, we get from (4.3): λ is in the resolvent set of H if and only if the boundary-value problem

$$(\sigma\partial_z + \beta - i\kappa\sigma_c + \chi(\lambda) - \lambda)\varphi = 0 \quad \text{with b.c. (4.1)} \tag{4.5}$$

has only the trivial solution $\varphi = 0$ in $\mathbb{H}^1([0, L]; \mathbb{C}^2)$. The transfer matrix corresponding to (4.5) is

$$T_k(z, \lambda) = \frac{e^{-\gamma_k z}}{2\gamma_k} \begin{pmatrix} \gamma_k + \mu_k + e^{2\gamma_k z}(\gamma_k - \mu_k) & i\kappa_k\left(1 - e^{2\gamma_k z}\right) \\ -i\kappa_k\left(1 - e^{2\gamma_k z}\right) & \gamma_k - \mu_k + e^{2\gamma_k z}(\gamma_k + \mu_k) \end{pmatrix} \tag{4.6}$$

for $z \in S_k$, where $\mu_k = \lambda - \chi_k(\lambda) - \beta_k$ and $\gamma_k = \sqrt{\mu_k^2 + \kappa_k^2}$ (see [2, 14]). The right-hand side of (4.6) does not depend on the branch of the square root in γ_k since the expression is even with respect to γ_k. Denote the overall transfer matrix of (4.5) by $T(z_1, z_2; \lambda)$ for $z_1, z_2 \in [0, L]$. The function

$$h(\lambda) = \begin{pmatrix} r_L, & -1 \end{pmatrix} T(L, 0; \lambda) \begin{pmatrix} r_0 \\ 1 \end{pmatrix} = \begin{pmatrix} r_L, & -1 \end{pmatrix} \prod_{k=m}^{1} T_k(l_k; \lambda) \begin{pmatrix} r_0 \\ 1 \end{pmatrix} \tag{4.7}$$

defined in $\mathbb{C} \setminus \mathcal{W}$ is the characteristic function of H: its roots are the eigenvalues of H.

$$\mathcal{R} := \{\lambda \in \mathbb{C} \setminus \mathcal{W} : h(\lambda) \neq 0\}$$

is the resolvent set. Consequently, all $\lambda \in \mathbb{C} \setminus \mathcal{W}$ are either eigenvalues of H or in $\mathcal{R}$, i.e., there is no essential (continuous or residual) spectrum in $\mathbb{C} \setminus \mathcal{W}$. We note that the relation

$$\max \operatorname{Re} \mathcal{W} \ll -1$$

holds for physically plausible parameter constellations.

The following lemma provides an approximate upper bound for the real parts of the eigenvalues.

Lemma 8. *Let $\lambda \in \mathbb{C} \setminus \mathcal{W}$ be in the point spectrum of H. Then λ is geometrically simple, and its real part satisfies the estimate*

$$\operatorname{Re}\lambda \leq \Lambda_u := \max_{k=1,\dots,m} \left\{ -\frac{\Gamma_k}{2}, \operatorname{Re}\beta_k + 2\rho_k \right\}.$$

It is useful to treat the operator H as a perturbation of the operator

$$H_0 = \begin{pmatrix} \sigma\partial_z + \beta & 0 \\ 0 & i\Omega_r - \Gamma \end{pmatrix}$$

defined in $Y \subset X$ (see also [14, 15]). The spectrum of H_0 consists of $\mathcal{W}$ and the sequence of simple eigenvalues

$$\lambda_j^0 := \frac{1}{L}\Big[\sum_{k=1}^{m} \beta_k l_k + \frac{1}{2}\log(r_0 r_L) + j\pi i\Big] \text{ for } j \in \mathbb{Z}.$$

The following theorem establishes how the growth properties of the semigroup $T(t)$ are related to the spectrum of H.

Theorem 9. *Let ξ_0 be defined by*

$$\xi_0 := \begin{cases} \max\{\operatorname{Re}\lambda_0^0, \max\operatorname{Re}\mathcal{W}\} & \text{if } r_0 r_L \neq 0, \\ \max\operatorname{Re}\mathcal{W} & \text{if } r_0 r_L = 0. \end{cases}$$

For $\xi > \xi_0$, there are at most finitely many eigenvalues of H of finite algebraic multiplicity in the right half-plane $\mathbb{C}_\xi := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq \xi\}$. Moreover, X can be decomposed into two $T(t)$-invariant subspaces

$$X = X_+ \oplus X_-,$$

where X_+ is at most finite-dimensional and spanned by the generalized eigenvectors associated to the eigenvalues of H in $\mathbb{C}_\xi$. There exists a constant M such that the restriction of $T(t)$ to X_- is bounded according to

$$\|T(t)|_{X_-}\| \leq M e^{\xi t} \tag{4.8}$$

in any norm which is equivalent to the X-norm.

Remark. The eigenvalues of H can be computed numerically by solving the complex equation $h(\lambda) = 0$. The eigenvalues of H_0 in $\mathbb{C} \setminus \mathcal{W}$ form the sequence λ_j^0 for $\kappa = 0$, $\rho = 0$, $r_0^0 r_L^0 \neq 0$. The roots of the characteristic function h can be obtained by continuing along the parameter path $\theta\kappa$, $\theta\rho$, $r_0^0 + \theta(r_0 - r_0^0)$, $r_L^0 + \theta(r_L - r_L^0)$ for $\theta \in [0, 1]$.

4.3. Existence and properties of the finite-dimensional center manifold. The preceding results permit the application of theorems about the persistence and properties of normally hyperbolic invariant manifolds in Banach spaces [7–9] to the semiflow $S(t, \cdot)$ generated by system (4.2), (4.4) under the following condition:

Assumption 10. *Assume there exist a number $\xi \in (\xi_0, 0)$ and a simple connected compact set $\mathcal{K} \subset \mathbb{R}^m$ such that for all $n \in \mathcal{K}$ the spectrum of $H(n)$ has the splitting*

$$\operatorname{spec} H(n) = \sigma_c(n) \cup \sigma_s(n),$$

where

$$\operatorname{Re}\sigma_c(n) = 0, \quad \operatorname{Re}\sigma_s(n) < \xi < 0.$$

Due to Theorem 9, the number of elements of $\sigma_c(n)$ is finite and, hence, constant in $\mathcal{K}$ if the eigenvalues are counted according to their algebraic multiplicity. We denote this number by q. Moreover, for each $\gamma \in [\xi, 0)$, there exists a bounded simple connected open set $U_\gamma \supset \mathcal{K}$ such that the splitting of $\operatorname{spec} H(n)$ can be extended to U_γ:

$$\operatorname{spec} H(n) = \sigma_c(n) \cup \sigma_s(n),$$

where for all $n \in U_\gamma$

$$\operatorname{Re}\sigma_c(n) > \gamma, \quad \operatorname{Re}\sigma_s(n) < \xi.$$

There exist spectral projections of $H(n)$, $P_c(n)$, and $P_s(n) \in \mathcal{L}(X)$ corresponding to this splitting. They are well defined and unique for all $n \in U_\xi$ and depend smoothly on n. We define the corresponding closed invariant subspaces of X by $X_c(n) = \operatorname{Im} P_c(n) = \ker P_s(n)$ and $X_s(n) = \operatorname{Im} P_s(n) = \ker P_c(n)$. The complex dimension of $X_c(n)$ is q. Let $B(n) : \mathbb{C}^q \to X$ be a basis of $X_c(n)$ which depends smoothly on n. $B(\cdot)$ is well defined in U_ξ. Using this notation, we can state the following theorem:

Theorem 11 (model reduction)**.** *Let $k > 2$ be any integer and $E_{\max} > 0$. Then there exist an $\varepsilon_0 > 0$ and an open neighborhood $U \subset U_\xi$ of $\mathcal{K}$ such that by using the sets*

$$\mathcal{B} := \{(E_c, n) \in \mathbb{C}^q \times \mathbb{R}^m : \|E_c\| < bE_{\max} + 1, n \in U\} \subset \mathbb{C}^q \times \mathbb{R}^m,$$
$$\mathcal{N} := \{(E, n) \in X \times \mathbb{R}^m : \|E\| < E_{\max}, n \in \Upsilon\} \subset X \times \mathbb{R}^m,$$

where b is defined by $b := \max_{n\in \operatorname{cl} U} \|B(n)^{-1}P_c(n)\|$ and Υ is an arbitrary closed subset of U, the following statements hold. For all $\varepsilon \in (0, \varepsilon_0)$, there exists a C^k manifold $\mathcal{C}$ satisfying:

(i) (Invariance) *$\mathcal{C}$ is $S(t, \cdot)$-invariant relative to $\mathcal{N}$.*

(ii) (Representation) *$\mathcal{C}$ can be represented as the graph of a map which maps*

$$(E_c, n, \varepsilon) \in \mathcal{B} \times (0, \varepsilon_0) \to ([B(n) + \varepsilon\nu(E_c, n, \varepsilon)]E_c, n) \in X \times \mathbb{R}^m,$$

where $\nu : \mathcal{B} \times (0, \varepsilon_0) \to \mathcal{L}(\mathbb{C}^q; X)$ is C^{k-2} with respect to all arguments.

(iii) (Exponential attraction) *Denote the E-component of $\mathcal{C}$ by*

$$E_X(E_c, n, \varepsilon) = [B(n) + \varepsilon\nu(E_c, n, \varepsilon)]E_c \in X.$$

Let (E, n) be such that $S(t; (E, n)) \in \mathcal{N}$ for all $t \geq 0$. Then there exist $(E_c, n_c) \in \mathcal{B}$, $M > 0$, and $t_c \geq 0$ such that

$$\|S(t + t_c; (E, n)) - S(t; (E_X(E_c, n_c, \varepsilon), n_c))\| \leq Me^{\xi t} \text{ for all } t \geq 0. \tag{4.9}$$

(iv) (Flow) *The values $\nu(E_c, n, \varepsilon)E_c$ are in Y and their $P_c(n)$-component is 0 for all $(E_c, n, \varepsilon) \in \mathcal{B} \times (0, \varepsilon_0)$. The flow on $\mathcal{C} \cap \mathcal{N}$ is differentiable with respect to t and governed by the following system of ordinary differential equations*:

$$\begin{aligned} \frac{dE_c}{dt} &= \left[H_c(n) + \varepsilon a_1(E_c, n, \varepsilon) + \varepsilon^2 a_2(E_c, n, \varepsilon)\nu(E_c, n, \varepsilon)\right] E_c, \\ \frac{dn}{dt} &= \varepsilon F(E_c, n, \varepsilon), \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} H_c(n) &= B(n)^{-1}H(n)P_c(n)B(n), \\ a_1(E_c, n, \varepsilon) &= -B(n)^{-1}P_c(n)\partial_n B(n)F(E_c, n, \varepsilon), \\ a_2(E_c, n, \varepsilon) &= B(n)^{-1}\partial_n P_c(n)F(E_c, n, \varepsilon)(Id - P_c(n)), \\ F(E_c, n, \varepsilon) &= (f_k(n_k) - g_k(n_k)[E_X(E_c, n_c, \varepsilon), E_X(E_c, n_c, \varepsilon)])_{k=1}^m. \end{aligned}$$

System (4.10) is symmetric with respect to rotation $E_c \to E_c e^{i\varphi}$, and ν satisfies the relation

$$\nu(e^{i\varphi}E_c, n, \varepsilon) = \nu(E_c, n, \varepsilon)$$

for all $\varphi \in [0, 2\pi)$.

5. Mode Analysis

The graph of the center manifold enters the description (4.10) of the flow on $\mathcal{C}$ only in the form $O(\varepsilon^2)\nu$. All other terms appearing in (4.10) can be expressed analytically as functions of the eigenvalues of $H(n)$. Systems of the form (4.10) but replacing ν by 0 are called *mode approximation models*. They have the form

$$\begin{aligned}\frac{dE_c}{dt} &= [H_c(n) + \varepsilon a_1(E_c, n, \varepsilon)]\, E_c,\\ \frac{dn}{dt} &= \varepsilon F(E_c, n, \varepsilon),\end{aligned} \tag{5.1}$$

where $E_c \in \mathbb{C}^q$, $n \in \mathbb{R}^m$, and

$$\begin{aligned} H_c(n) &= B(n)^{-1}H(n)P_c(n)B(n),\\ a_1(E_c, n, \varepsilon) &= -B(n)^{-1}P_c(n)\partial_n B(n)F(E_c, n, \varepsilon),\\ F(E_c, n, \varepsilon) &= (f_k(n_k) - g_k(n_k)[B(n)E_c, B(n)E_c])_{k=1}^m\,. \end{aligned}$$

These models are implicit systems of ordinary differential equations because the eigenvalues of H are given only implicitly as roots of the characteristic function h of H. The consideration of mode approximations has proven to be extremely useful for numerical and analytical investigations of longitudinal effects in multisection semiconductor lasers because the dimension of system (4.10) is typically low (q is often either 1 or 2); see, e.g., [2–4, 6, 16, 21, 23, 24].

5.1. Delayed optical feedback. We demonstrate the use of system (5.1) for the classical experiment of a stationary single-mode laser which is subject to delayed optical feedback [22]. We investigate a two-

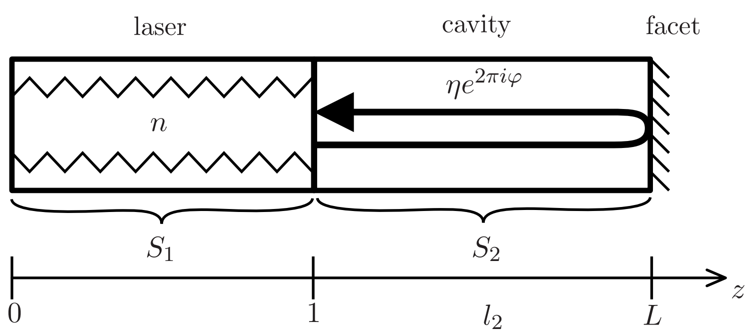

Fig. 2. Sketch of a two-section laser resembling a delayed feedback experiment. The section lengths are 220 μm and 250 μm in the original physical dimensions.

section laser, where S_1 is a single-mode DFB laser and S_2 is a passive waveguide providing delayed optical feedback from its facet (see Fig. 2), and, hence, representing an extremely short optical cavity. In this case, S_2 is passive, i.e., G_2 and ρ_2 are identically zero. Hence, n_2 does not couple into system (2.1)–(2.4) such that we can consider $n(t) = n_1(t)$ as a scalar. Our primary bifurcation parameters are the strength η and the phase φ of the reflectivity at the facet $r_L = \eta e^{2\pi i\varphi}$.

There is a parameter point (φ_0, η_0) where a stationary state ($E_c(t) = E_{c,0}e^{\lambda_1 t}, n(t) = n_0 = \text{const}$) exists such that λ_1 is on the imaginary axis and of double algebraic multiplicity and all other eigenvalues of $H(n_0)$ have negative real parts. It turns out that the dynamics of (4.2), (4.4) is described by an unfolding of this degeneracy in a large part of the parameter plane (φ, η), i.e., we have $q = 2$. We denote the critical eigenvalues of H in the vicinity of the degeneracy by λ_1 and λ_2 and the corresponding eigenvectors by v_1 and v_2 (scaled such that the ψ-component of $v_{1,2} = (r_0, 1)$). Then, we can choose $[(v_1 - v_2)/(\lambda_1 - \lambda_2), (v_1 + v_2)/2]$ as the basis B of the critical subspace and the corresponding adjoints as

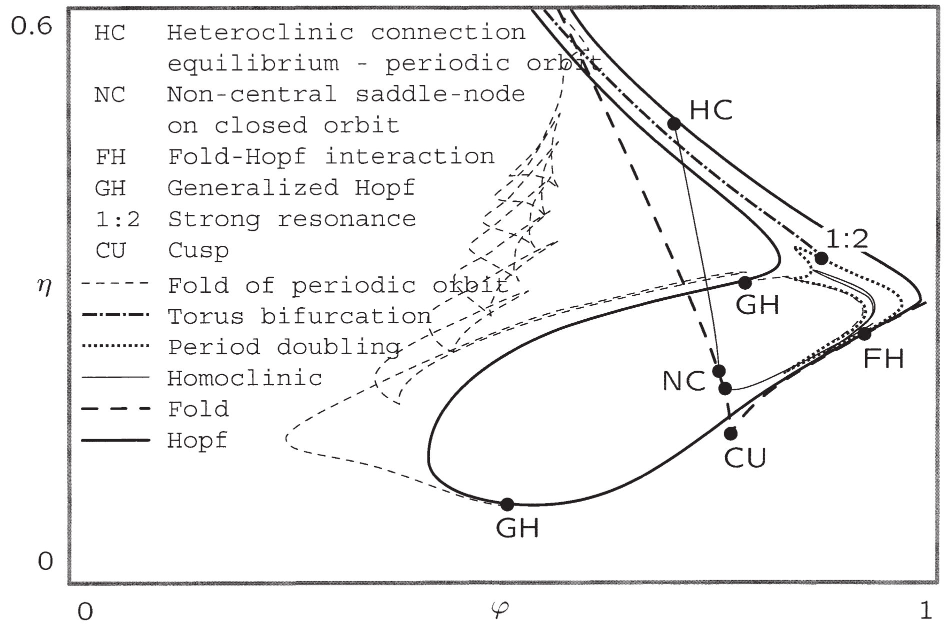

Fig. 3. Bifurcation diagram for the two-mode approximation (5.1) (with $q = 2$) in the parameter plane (φ, η) (see [16] for the particular parameter values).

spectral projection $B^{-1}P_c$ in (5.1). Hence, system (5.1) has a real dimension of 4 after reduction of the rotational symmetry in E_c (see [16] for details).

Numerical continuation of the equilibria and periodic orbits of the reduced system (corresponding to rotating waves and modulated rotating waves in the original system) reveals the bifurcation diagram in Fig. 3.

Remark. The diagram in Fig. 3 is incomplete because of the complex dynamics in the vicinity of some of the bifurcations (e.g., fold-Hopf interaction, 1:2 resonance, homoclinics to saddle-focus). However, it describes and locates some phenomena which are of great interest for applications, e.g., oscillations or excitability.

The third author was supported by the Sonderforschungsbereich 555 "Komplexe Nichtlineare Prozesse" of the Deutsche Forschungsgemeinschaft.

REFERENCES

1. E. A. Avrutin, J. H. Marsh, and J. M. Arnold, "Modelling of semiconductor laser structures for passive harmonic mode locking at terahertz frequencies," *Int. J. Optoelectronics,* **10**, 427–432 (1995).
2. U. Bandelow, *Theorie longitudinaler effekte in* 1.55 μm *mehrsektions DFB-laserdioden*, PhD thesis, Humboldt-Universität Berlin, (1994).
3. U. Bandelow, M. Radziunas, V. Tronciu, H.-J. Wünsche, and F. Henneberger, "Tailoring the dynamics of diode lasers by dispersive reflectors," In: *Proc. SPIE*, Vol. 3944 (2000), pp. 536–545.
4. U. Bandelow, L. Recke, and B. Sandstede, "Frequency regions for forced locking of self-pulsating multi-section DFB lasers," *Opt. Comm.,* **147**, 212–218 (1998).
5. U. Bandelow, M. Wolfrum, M. Radziunas, and J. Sieber, "Impact of gain dispersion on the spatio-temporal dynamics of multisection lasers," *IEEE J. Quant. El.,* **37**, 183–189 (2001).
6. U. Bandelow, H. J. Wünsche, B. Sartorius, and M. Möhrle, "Dispersive self Q-switching in DFB-lasers: Theory versus experiment," *IEEE J. Selected Topics Quant. El.,* **3**, 270–278 (1997).

7. P. W. Bates, K. Lu, and C. Zeng, "Existence and persistence of invariant manifolds for semiflows in Banach spaces," *Mem. Am. Math. Soc.,* **645**, 1–129 (1998).
8. P. W. Bates, K. Lu, and C. Zeng, "Persistence of overflowing manifolds for semiflow," *Commun. Pure Appl. Math.,* **52**, No. 8, 983–1046 (1999).
9. P. W. Bates, K. Lu, and C. Zeng, "Invariant foliations near normally hyperbolic invariant manifolds for semiflows," *Trans. Am. Math. Soc.,* **352**, 4641–4676 (2000).
10. O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H. O. Walther, *Delay Equations: Functional–, Complex–, and Nonlinear Analysis*, Series: Appl. Math. Sci., **110**, Springer, New York (1995).
11. D. Marcenac, *Fundamentals of laser modelling*, Ph.D. thesis, University of Cambridge (1993).
12. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, Springer-Verlag, New York (1983).
13. M. Radziunas, H.-J. Wünsche, B. Sartorius, O. Brox, D. Hoffmann, K. Schneider, and D. Marcenac, "Modeling self-pulsating DFB lasers with integrated phase tuning section," *IEEE J. Quant. El.,* **36**, 1026–1034 (2000).
14. L. Recke, K. Schneider, and V. Strygin, "Spectral properties of coupled wave equations," *Z. Angew. Math. Phys.,* **50**, 923–933 (1999).
15. J. Rehberg, H.-J. Wünsche, U. Bandelow, and H. Wenzel, "Spectral properties of a system describing fast pulsating DFB lasers," *ZAMM, Z. Angew. Math. Mech.,* **77**, 75–77 (1997).
16. J. Sieber, "Numerical bifurcation analysis for multi-section semiconductor lasers," *SIAM J. Appl. Dyn. Sys.,* **1**, No. 2, 248–270 (2002)
17. J. Sieber, *Longtime behavior of the traveling-wave model for semiconductor lasers*, Preprint 743, WIAS (2002).
18. J. Sieber, U. Bandelow, H. Wenzel, M. Wolfrum, and H.-J. Wünsche, *Travelling wave equations for semiconductor lasers with gain dispersion*, Preprint 459, WIAS (1998).
19. H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam–New York (1978).
20. B. Tromborg, H. E. Lassen, and H. Olesen, "Travelling wave analysis of semiconductor lasers," *IEEE J. Quant. El.,* **30**, 939–956 (1994).
21. V. Tronciu, H.-J. Wünsche, J. Sieber, K. Schneider, and F. Henneberger, "Dynamics of single mode semiconductor lasers with passive dispersive reflectors," *Opt. Comm.,* **182**, 221–228 (2000).
22. G. H. M. van Tartwijk and G. P. Agrawal, "Laser instabilites: A modern perspective," *Prog. Quant. El.,* **22**, 43–122 (1998).
23. H. Wenzel, U. Bandelow, H.-J. Wünsche, and J. Rehberg, "Mechanisms of fast self pulsations in two-section DFB lasers," *IEEE J. Quant. El.,* **32**, 69–79 (1996).
24. H. J. Wünsche, O. Brox, M. Radziunas, and F. Henneberger, "Excitability of a semiconductor laser by a two-mode homoclinic bifurcation," *Phys. Rev. Lett.,* **88** (2002).

Jan Sieber
Weierstrass Institute for Applied Analysis and Stochastics,
D-10117 Berlin, Germany
E-mail: `sieber@wias-berlin.de`
Lutz Recke
Institute of Mathematics, Humboldt University of Berlin,
D-10099 Berlin, Germany
E-mail: `recke@mathematik.hu-berlin.de`
Klaus R. Schneider
Weierstrass Institute for Applied Analysis and Stochastics,
D-10117 Berlin, Germany
E-mail: `schneider@wias-berlin.de`